Aristotle University of Thessaloniki

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Master's Thesis

Sheaves on Topological Spaces, Sites and Group Categories

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Chapter 1

Introduction

1.0.1 Historical background

The concept of a sheaf has its roots in the study of analytic continuation of functions, which began in the 19th century and was subsequently formalized by Hermann Weyl in his notable work on the "idea" of the Riemann surface. Later studying domains of holomorphy and the Cousin problems, Henri Cartan and Kiyoshi Oka explored the notion of ideals on a domain, which, as it turned out, are effectively sheaves. In 1944, Cartan referred to them as "coherent systems of punctual ideals," while Oka, in 1949, addressed "ideals with an indeterminate domain."

Post-World War II, Jean Leray contributed significantly by providing the first comprehensive and explicit definition of a sheaf on a space, described with closed sets of that space. Cartan, building upon this idea, reformulated the definition of sheaves in terms of open sets during his seminars in 1948-49 and 1950-51. In these seminars, Lazard also introduced the equivalent definition of a sheaf on a space as an étale bundle into this space. This profound equivalence between these two notion of sheaves became a central motivating force in a new area of mathematics, topos theory.

At this time, sheaves could be perceived as continuous mappings of the open sets of a topological space X to another topological space, which is constructed by the images of the mappings, such that they agree on the intersections. The final space can be projected on the space X in a continuous and locally homeomorphic way.

Subsequently, J. P. Serre and other mathematicians found application of sheaves that extended beyond topology, in algebraic geometry. Although the étale bundle approach to the sheaf construction gives continuous maps by solely defined on subsets of X, they could be also defined on objects U, which need not be subsets contained within X, but rather mappings from some other space U into X. This expansion of scope introduced concepts from category theory.

This shift prompted Grothendieck to redefine sheaves within a broader context. He replaced the notion of a partially ordered collection of open subsets of a space with objects from a category $\mathcal{C}$. In this category, families of maps with codomain an object X replaced the notion of "covers" of an open subject. Under this structure on a category, known as "Grothendieck topology", a sheaf transformed into a functor that could be collated over each cover.

After that, Grothendieck's mathematical landscape was transformed. For him, topology became an exploration of (the cohomology of) sheaves, and the sheaves situated within a particular

Grothendieck topology formed a topos, later known as a Grothendieck topos. In this framework, the formulation of various cohomologies theories set the stage for an extensive effort to tackle the Weyl conjectures regarding solutions to polynomial equations. In the early 1960s, these highly versatile ideas underwent rapid development, driven by the collaborative efforts of A. Grothendieck and his colleagues, including J. L. Verdier, M. Artin, M. Giraud, M. Hakim, L. Illusie, and others. These groundbreaking concepts were documented in an expanded three-volume publication titled "SGA IV", totaling 1623 pages. Their far-reaching impact resonated throughout the domain of algebraic geometry and ultimately played a pivotal role in the resolution of the Weyl conjectures by P. Deligne in 1974.

1.0.2 Thesis structure

In this master thesis, we make an exploration of sheaf theory, focusing on its foundational principles in topology and category theory and showcasing some applications in the form of equivalences of sheaves with other structures. The aim of this work is to present this profound area of mathematics with precision and clarity, providing a comprehensive understanding of this intricate subject.

We begin in the first chapter with an introduction of sheaf of sets on a topological space. After presenting the basic concepts regarding sheaves, we establish a fundamental equivalence between presheaves and bundles on spaces, which is restricted to an equivalence between sheaves and étale bundles. Next, we see how the sheaves of sets can be equipped with algebraic structure, bringing about sheaves of abelian groups, rings, modules, etc. We conclude with two sections, one about the important notion of inverse image and one showing that the category of sheaves is a topos. This chapter lays the groundwork for our understanding of sheaf theory's fundamental concepts, providing a solid foundation for the subsequent chapters.

In the second chapter, we expand the notion of sheaves to sheaves on categories with a specific structure, called sites. This extension broadens the scope of sheaf theory, enabling its application in various mathematical domains. We introduce Grothendieck topologies and how they can be used for the definition of sheaves on a site, concluding with the two-step process of sheafification, a significant contruction for transforming presheaves into sheaves.

In the final chapter, building upon the basic theory established in the previous chapters, we demonstrate an application in the domain of group representations; we show the equivalence between set representations of a group and sheaves on its orbit category. For this purpose, we utilise a generalisation of continuous functions, continuous and cocontinuous functors of sites, as well as geometric morphisms of topoi, the atomic topology and the transporter category. This chapter showcases the power of generalising all these concepts from topology, geometry and algebra, and processing them in the light of topos theory.

Chapter 2

Sheaves on Topological Spaces

2.1 Sheaves of Sets

Initially, we introduce some notation. We typically denote topological spaces with the letters X, Y and an arbitrary category with $\mathcal{C}$. We put some specific categories in bold, for example **Set** is the category of sets. With $\mathcal{O}(X)$ we denote the category of open sets of a topological space X and exponential of categories is the category of functors from the exponent to the base. So, the category $\mathbf{Set}^{\mathcal{O}(X)^{\mathsf{op}}}$ is the category of the functors $\mathcal{O}(X)^{\mathsf{op}} \to \mathbf{Set}$, which are called presheaves. Finally, if we have a presheaf $P\colon \mathcal{O}(X)^{\mathsf{op}} \to \mathbf{Set}$, open sets $W \subset U$ and an element f of the set $P(U)$, then we denote with $f|_W$ the restriction of f which belongs to the set $P(W)$ or in other words, it is the image of f under the morphism $P(i)$ where $i : W \hookrightarrow U$ is the inclusion morphism in the category $\mathcal{O}(X)$.

Definition 2.1.1 (Sheaf on topological space)**.** Let X be a topological space. A **sheaf of sets** F on X is a presheaf on X, such that each open covering $\{U_i\}_{i\in I}$ of an open set U ($U = \bigcup_{i\in I} U_i$) yields an equaliser diagram:

$$F(U) \overset{e}{\dashrightarrow} \prod_i F(U_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j), \tag{2.1}$$

where for $f \in F(U)$, $e(f) = \{f|_{U_i} : i \in I\}$ and for a family $\{f_i \in C(U_i)\}_{i\in I}$,

$$p_1(\{f_i\}) = \{f_i|_{U_i\cap U_j}\}, \quad p_2(\{f_i\}) = \{f_j|_{U_i\cap U_j}\}.$$

We recall that equaliser is the object E with the morphism e, in our case in the category of sets, such that $p_1 \circ e = p_2 \circ e$, and is universal, in the sense that if we have another object E' with morphism e', such that $p_1 \circ e' = p_2 \circ e'$, then there is a morphism $d\colon E' \to E$, such that $e' = e \circ d$.

We can define the morphisms $F \to G$ of sheaves as the natural transformations of functors and as a result we can define the category of sheaves of sets on X, denoted by $\mathbf{Sh}(X)$. Apparently, $\mathbf{Sh}(X)$ is a full subcategory of the category of presheaves.

Next, we present the first fact about sheaves.

Proposition 2.1.2. *Every sheaf must send the empty set onto a one-point set.*

Proof. For any space X and any sheaf F, the empty set has an empty cover. However, the product over an empty index set is a one-point set, so the equaliser in the definition of sheaf becomes:

$$F(\emptyset) \longrightarrow \{*\} \rightrightarrows \{*\},$$

which gives us $F(\emptyset) = \{*\}$, as wanted. □

Definition 2.1.3 (Subsheaf)**.** For a sheaf F on a topological space X, a **subsheaf** of F is a subfunctor of F, which is itself a sheaf. We can define the category of subsheaves of F, having a morphism from subsheaf F_1 to subsheaf F_2, if F_1 is a subsheaf of F_2. We denote this category $\mathsf{Sub}_{\mathbf{Sh}(X)}(F)$.

Proposition 2.1.4. *For a topological space X and a sheaf F on X, the following are equivalent:*

1. *a subfunctor S of F is a subsheaf,*
2. *for every open set $U \subset X$, a covering $U = \bigcup_{i\in I} U_i$ and an element $f \in F(U)$, we have $f \in S(U)$ if and only if $f|_{U_i} \in S(U_i)$ for all $i \in I$.*

Proof. (2) ⇒ (1): From this condition, we have that S is a sheaf. It is also a subfunctor of F, thus a subsheaf.

(1) ⇒ (2): We need to prove that for every open set $U \subset X$, a covering $U = \bigcup_{i\in I} U_i$ and an element $f \in F(U)$, $f \in S(U)$ implies that $f|_{U_i} \in S(U_i)$ for all $i \in I$.

Since S is a subfunctor of F, there is a monic map $m\colon S \to F$. We have an element $f \in F(U)$ with $f|_{U_i} \in S(U_i)$ for every U_i. Equivalently we can say that there are $g|_{U_i} \in S(U_i)$, such that $m(g|_{U_i}) = f|_{U_i}$. We claim that $S(U)$ is the fiber product in the pullaback square below and then we would have that there is an element g with $m(g) = f$ or equivalently $f \in S(U)$, as desired.

$$\begin{array}{ccc} S(U) & \xrightarrow{e'} & \prod S(U_i) \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ F(U) & \xrightarrow{e} & \prod F(U_i) \end{array}$$

To prove this claim we start with the two equaliser diagrams for the sheaves S and F and we construct the diagram:

$$\begin{array}{ccccc} S(U) & \xrightarrow{e'} & \prod S(U_i) & \overset{p_1'}{\underset{p_2'}{\rightrightarrows}} & \prod S(U_i \cap U_j) \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ F(U) & \xrightarrow{e} & \prod F(U_i) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \prod F(U_i \cap U_j) \end{array}$$

where the vertical maps are the induced monic maps from m. Since m is a natural transformation, the diagram is commutative.

Assuming we have an arbitrary set A with $d_1\colon A \to \prod S(U_i)$ and $d_2\colon A \to F(U)$, such that $e \circ d_2 = m \circ d_1$. Then, from the commutativity and the equalisers, we have

$$m \circ p_1' \circ d_1 = p_1 \circ m \circ d_1 = p_1 \circ e \circ d_2 = p_2 \circ e \circ d_2 = p_2 \circ m \circ d_1 = m \circ p_2' \circ d_1.$$

Since m is monic, we get $p'_1 \circ d_1 = p'_2 \circ d_1$, so from the equaliser, there is a unique function $d\colon A \to S(U)$ with $d_1 = e' \circ d$. Also, $e \circ m \circ d = m \circ e' \circ d = m \circ d_1 = e \circ d_2$ and since e is equaliser, so monic, $d_2 = m \circ d$. Therefore, $S(U)$ is the fibered product, as desired. □

Definition 2.1.5 ($\mathbb{1}$)**.** For a topological space X, we define the constant sheaf $\mathbb{1}$ to be the functor that maps each open $U \subset X$ to the one point set, $\{*\}$, which is the terminal object of the category **Set**.

It is clear that $\mathbb{1}$ is a sheaf and it is also the terminal element in $\mathbf{Sh}(X)$.

Proposition 2.1.6. *Consider a topological space X. Then we have the equivalence of categories:*

$$\mathcal{O}(X) \cong \mathsf{Sub}_{\mathbf{Sh}(X)}(\mathbb{1}).$$

Proof. For an arbitrary open set $W \subset X$, we define the presheaf S_W, such that for open $U \subset X$, $S_W(U) = \{*\}$ if $U \subset W$ and $S_W(U) = \emptyset$, otherwise (in fact this is the Yoneda embedding $\mathbf{y}(W)$ mentioned below). It is easy to see that this is a sheaf.

Conversely, consider S to be a subsheaf of $\mathbb{1}$. Then, each $S(U)$ is either $\{*\}$ or $\emptyset$, but for $V \subset U$, if $S(U) = \{*\}$, then $S(V) = \{*\}$, as well. Also, by the equaliser condition, if we have an open cover of U, $\bigcup_i U_i = U$ and $S(U_i) = \{*\}$ for all i, then $S(U) = \{*\}$. Therefore, we can get the set

$$W = \bigcup \{U \in \mathcal{O}(X) \mid S(U) = \{*\}\}.$$

It is easy to see that $S(U) = \{*\}$ if and only if $U \subset W$. But this is exactly the definition of S_W, so $S = S_W$.

We conclude that the desired bijection $\mathcal{O}(X) \cong \mathsf{Sub}_{\mathbf{Sh}(X)}(\mathbf{1})$ is $W \mapsto S_W$. If we see them as partially ordered sets (or categories with morphisms being the inclusions), then the order is clearly preserved, so it is an isomorphism (or equivalence of categories). □

This result is remarkable, because it shows that one can recover the partially ordered set of open subsets of a topological space X from the category $\mathbf{Sh}(X)$; it is the set of the subobjects of the terminal object $\mathbb{1}$. Hence, we can say that the category of sheaves of sets on X determines the topology of X.

2.2 Sieves and Sheaves

On a topological space X and given an open set U, we can define a presheaf of the form $\mathsf{Hom}(-, U)$, which is defined:

$$\mathsf{Hom}(-,U)(V) = \mathsf{Hom}(V,U) = \begin{cases} \{*\} & \mathsf{if} V \subset U \\ \emptyset & \mathsf{otherwise.} \end{cases}$$

This is trivially a sheaf and it is exactly the **representative presheaf** or the **Yoneda embedding**, $\mathbf{y}(U)$, on the category $\mathcal{O}(X)^{\mathsf{op}}$.

We now define a **sieve** S on $\mathcal{O}(U)$ as the subset of open sets in $\mathcal{O}(U)$ with the property that if $W \subset V \in S$, then $W \in S$ (in the next chapter, we give the general definition). Each family of open sets $\{V_i\}_{i\in I}$ can generate a sieve, which is the set of all open $V \in \mathcal{O}(U)$ such that $\exists i$ with $V \subset V_i$. If the family has only one element, then we call that sieve, a **principal sieve**. Furthermore, if the union of all sets in a sieve is the original set U, then we call that a **covering sieve**.

We can make a new definition of sheaves, replacing open coverings with covering sieves, which has the advantage that we describe sheaves only in terms of objects in the category $\mathbf{PSh}(X) = \mathbf{Set}^{\mathcal{O}(X)^{op}}$. We will see how this alternative definition is useful in the next chapter (Definition 3.1.1).

For the next proposition, we use the fact that a sieve is equivalent with a subfunctor of the Yoneda embedding. We prove this result for the generalised case in Proposition 3.1.2.

Proposition 2.2.1. *Let X be a topological space and P a presheaf on X. Then P is a sheaf if and only if, for every open set U and every covering sieve S on U, we have an isomorphism* $\mathsf{Hom}(\boldsymbol{y}(U), P) \cong \mathsf{Hom}(S, P)$, *which is induced by the inclusion of functors $i\colon S \to \boldsymbol{y}(U)$.*

Proof. At first, we take the following equaliser diagram for any covering $U = \bigcup_{i\in I} U_i$, considering that P is just a presheaf.

$$E \xrightarrow{d} \prod_i P(U_i) \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{i,j} P(U_i \cap U_j), \tag{2.2}$$

Since we have sets, it is easy to see that E consists of the families of elements $\{x_i \in P(U_i)\}_{i\in I}$ such that $x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j}$ for all pairs (i, j) (because these two elements must be equivalent). We can change the sets U_i with the sieve S, which is generated by $\{U_i\}_{i\in I}$. For any $V \in S$, we define $x_V = x_i|_V$ if $V \subset U_i$. This choice is independent of i, since x_i are the elements where all sets U_i agree on the intersections. Therefore, E is described equivalently by the families of elements $\{x|_V \in P(V)\}_{V\in S}$, where for $V' \subset V$, $x_V|_{V'} = x|_{V'}$.

We can see the sieve S as a presheaf on X, where $S(V) = \{*\}$ if $V \in S$ and $S(V) = \emptyset$, otherwise. Then, each element $x_V \in P(V)$ is a natural transformation $x_V\colon S(V) \to P(V)$ with $* \mapsto x_V$ if $V \in S$. If $V' \subset V$, then we have the diagram

$$\begin{array}{ccc} S(V) & \xrightarrow{x_V} & P(V) \\ \downarrow & & \downarrow \\ S(V') & \xrightarrow{x_{V'}} & P(V'), \end{array}$$

which commutes, since $x_V|_{V'} = x|_{V'}$. So, a family of elements $\{x_V\}_{V\in S}$ is actually a natural transformation $S \to P$. Hence, the equaliser E is the set of all natural transformations $\theta\colon S \to P$ or, in other words, the set $\mathsf{Hom}(S, P)$.

Next, the map $i\colon S \to \mathbf{y}(U)$ induces the map $i_*\colon \mathsf{Hom}(\mathbf{y}(U), P) \to \mathsf{Hom}(S, P)$ and we consider the isomorphism $\mathsf{Hom}(\mathbf{y}(U), P) \cong P(U)$ given by the Yoneda Lemma. We also get the map $e\colon P(U) \to \prod P(U_i)$ with $x \mapsto \{x|_{U_i} \mid i \in I\}$. Therefore, we have the diagram:

$$\begin{array}{ccccc} \mathsf{Hom}(S,P) & \xrightarrow{d} & \prod P(U_i) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \prod_{i,j} P(U_i \cap U_j) \\ \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle e} & & \\ \mathsf{Hom}(\mathbf{y}(U),P) & \xrightarrow{\cong} & P(U) & & \end{array}$$

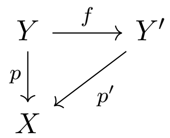
This diagram commutes. Indeed, consider an element $x \in PU$, which is mapped to $\{x|_{U_i} \mid i \in I\}$ through e. In the other direction, x can be considered as a natural transformation $\mathbf{y}(U) \to P$, where $x|_V : * \mapsto x_V \in P(V)$ for any $V \subset U$, which is also a natural transformation $S \to P$, since if $V \in S$ then $V \subset U$, so the map x_V is always defined. This natural transformation generates the maps $x_{U_i} : S(U_i) \to P(U_i)$ for any $i \in I$, so going through d, we get again the family $\{x|_{U_i} \mid i \in I\}$.

In this diagram, if i_* is an isomorphism, then $P(U)$ with the map e is an equaliser, thus P is a sheaf. Conversely, if P is a sheaf, then for any covering sieve S, the map i_* must be an isomorphism, because the equaliser is universal. $\square$

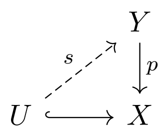
2.3 Equivalence with Bundles

2.3.1 Bundles

Definition 2.3.1 (Bundle)**.** For a topological space X, a continuous map $p\colon Y \to X$ is called a **bundle** over X. The bundles over X form a category $\mathbf{Bund}(X)$ with morphisms $f\colon p \to p'$ being continuous maps $f\colon Y \to Y'$ with $p' \circ f = p$.



Definition 2.3.2 (Cross-sections)**.** Let U be an open set of a topological space X and $p\colon Y \to X$ a bundle over X. A **cross-section** of the bundle p over U is a continuous map $s\colon U \to Y$ such that the following diagram commutes:



Let $\Gamma_p(U)$ be the sets of all cross-sections of the bundle p over U:

$$\Gamma_p(U) = \{s \mid s\colon U \to Y, p \circ s = i\colon U \hookrightarrow X\}.$$

Proposition 2.3.3 (Γ Functor)**.** *Γ_p is a sheaf and $\Gamma\colon \mathbf{Bund}(X) \to \mathbf{Sh}(X)$ is a functor from bundles to sheaves.*

Proof. Let $V \subseteq U$ be two open sets of X. Then, we can define a morphism $\Gamma_p(U) \to \Gamma_p(V)$ as the restriction operation. So, Γ_p is a presheaf on X ($\mathcal{O}(X)^{op} \to \mathbf{Set}$). Also if we have an open covering $\{U_i\}_{i \in I}$ of U and a family of cross-sections $\{s_i\colon U_i \to Y\}_{i \in I}$ that match on all the overlaps $U_i \cap U_j$, then "glueing" them together we get a cross-section s ($s(x) = s_i(x)$ if $x \in U_i$, thus $p \circ s(x) = p \circ s_i(x) = x$). Hence, we get that Γ_p is a sheaf. We call Γ_p the sheaf of cross-sections on the bundle p.

We can thus define a mapping $\Gamma\colon p \mapsto \Gamma_p$ from the category of bundles to the category of sheaves. If we have a map of bundles $f\colon p \to p'$, then $\Gamma(f)\colon \Gamma_p \to \Gamma_{p'}$ is the induced map defined as $\Gamma(f)|_U\colon \Gamma_p(U) \to \Gamma_{p'}(U)$ with $s \mapsto f \circ s$. We see that since $p' \circ f = p$, then $p' \circ \Gamma(f)|_U(s) = p' \circ f \circ s = p \circ s = i$, so $\Gamma(f)|_U(s) \in \Gamma_{p'}(U)$. $\square$

Definition 2.3.4 (Germs and Stalk)**.** Let P be a presheaf on a space X, a point $x \in X$, two open neighbourhoods U and V of x and two elements $s \in P(U)$, $t \in P(V)$. Then, s and t have the same germ at x, if there is an open neighbourhood W of x with $W \in U \cap V$, such that $s|_W = t|_W \in P(W)$. This defines an equivalence relation and we call the equivalence class of s under this relation, the **germ** of s at x denoted by $\mathsf{germ}_x s$. The set of all germs at x is called the **stalk** of P at x:

$$P_x = \{\mathsf{germ}_x s \mid s \in P(U), x \in U \mathsf{openin} X\}.$$

Remark 2.3.5. We can think of the stalk of P at x as a colimit. Consider $P^{(x)}$ to be the restriction of P to the sets containing x, then we have the functions $\mathsf{germ}_x \colon P(U) \to P_x$, which form a cone on $P^{(x)}$, because for $s \in P(U)$ and $W \subset U$, $\mathsf{germ}_x s = \mathsf{germ}_x s|_W$:

$$\begin{array}{ccc} P(U) & \longrightarrow & P(W) \\ & {}_{\mathsf{germ}_x}\searrow \quad \swarrow_{\mathsf{germ}_x} & \\ & P_x & \end{array}$$

This diagram is commutative, thus getting the aforementioned cone. If we have another cone $\{\tau_U \colon P(U) \to L\}_{x \in U}$, then from the definition of germ, we can define a unique function $t \colon P_x \to L$ with $t \circ \mathsf{germ}_x = \tau$. Therefore P_x is the colimit of the functor $P^{(x)}$.

Definition 2.3.6 (Λ Functor)**.** Let P be a presheaf on a space X and consider for every point $x \in X$, its stalk P_x. Then the disjoint union of these stalks form a set

$$\Lambda(P) = \coprod_{x \in X} P_x = \{\mathsf{germ}_x s \mid x \in X, U \in \mathcal{O}(X), s \in P(U)\}.$$

We define the function $p \colon \Lambda(P) \to X$ such that $\mathsf{germ}_x s \mapsto x$. Also, for any $s \in P(U)$, we define the function $\dot{s} \colon U \to \Lambda(P)$ such that $\dot{s}(x) = \mathsf{germ}_x s$ for $x \in U$. Then $\dot{s}$ is a section of p.

Remark 2.3.7. We can topologise the set $\Lambda(P)$. We define the base of open sets as all the image sets $\dot{s}(U) \subset \Lambda(P)$. For any element $\mathsf{germ}_x s$, there is some $U \in \mathcal{O}(X)$, such that $x \in U$, thus $\mathsf{germ}_x s \in \dot{s}(U)$. Furthermore, if $g \in \dot{s}(U) \cap \dot{t}(V)$, then there is a point $x \in U \cap V$, $s \in P(U)$, $t \in P(V)$, such that $g = \mathsf{germ}_x s = \mathsf{germ}_x t$. Hence, there is some open set $W \subset U \cap V$ with $x \in W$ and $s|_W = t|_W = r \in P(W)$, so $g = \mathsf{germ}_x r$ and $g \in \dot{r}(W) \subset \dot{s}(U) \cap \dot{t}(W)$.

With this topology on $\Lambda(P)$, the function p is a continuous map, since for an open $U \subset X$,

$$p^{-1}(U) = \bigcup_{\substack{V \subset U, \\ V \in \mathcal{O}(X)}} \bigcup_{s \in P(V)} \dot{s}(V).$$

So, $p \colon \Lambda(P) \to X$ is a bundle. Also, each $\dot{s}$ is continuous, because considering a base open set $\dot{t}(V) \subset \dot{s}(U)$, we have $V \subset U$ and $\forall x \in V$, $\mathsf{germ}_x t = \mathsf{germ}_x s$, thus $\dot{s}^{-1}(\dot{t}(V)) = V$. $\dot{s}$ is trivially an open map and a bijection, as well, therefore it is a homeomorphism $U \to \dot{s}(U)$. Considering this, each point $\mathsf{germ}_x s$ has the open neighbourhood $\dot{s}(U)$, which is mapped homeomorphically to U. This proves that p is a local homeomorphism. Finally, $\dot{s}$ is a cross-section of the bundle $p \colon \Lambda(P) \to X$.

Proposition 2.3.8. $\Lambda \colon \mathbf{PSh}(X) \to \mathbf{Bund}(X)$ *is a functor.*

It is easy to observe that Λ is indeed a functor.

Next, for a given presheaf P, we consider the sheaf $\Gamma\Lambda_P$ of the cross-sections of the bundle $p\colon \Lambda(P) \to X$ and for each open set U of X the functions:

$$\eta_U\colon P(U) \to \Gamma\Lambda_P(U), \qquad s \mapsto \eta_U(s) = \dot{s}.$$

Then, for $V \subset U$, we get the following commutative diagram

$$\begin{array}{ccc} P(U) & \xrightarrow{\eta_U} & \Gamma\Lambda_P(U) \\ \downarrow & & \downarrow \\ P(V) & \xrightarrow{\eta_V} & \Gamma\Lambda_P(V) \end{array} \qquad \begin{array}{ccc} s & \xmapsto{\eta_U} & \dot{s}\colon U \to \Lambda_P \\ \downarrow & & \downarrow \\ s|_V & \xmapsto{\eta_V} & \dot{s}|_V\colon V \to \Lambda_P \end{array}$$

and therefore $\eta\colon P \to \Gamma\Lambda_P$ is a natural transformation of presheaves.

The next result shows that every sheaf is a sheaf of cross-sections.

Theorem 2.3.9. *If the presheaf P is a sheaf, then the natural transformation η with $\eta_U(s) = \dot{s}$ is a natural isomorphism of functors: $P \cong \Gamma\Lambda_P$.*

Proof. We will show that η_U is a bijection.

For the injection part, we consider $s, t \in P(U)$, such that $\dot{s} = \dot{t}$. That means that $\forall x \in U$, $\mathsf{germ}_x s = \mathsf{germ}_x t$, thus there is an open $V_x \subset U$ with $s|_{V_x} = t|_{V_x}$. However, $\{V_x\}_{x\in U}$ is an open cover of U, so we consider the morphism:

$$PU \longrightarrow \prod_{x\in U} P(V_x),$$

which is an equaliser, thus it is a monomorphism. Since, s and t have the same images in $\prod_{x\in U} P(V_x)$, we get $s = t$.

For the surjection part, we consider a cross-section $t\colon U \to \Lambda(P)$.

$$\begin{array}{ccc} & & \Lambda(P) \\ & \overset{t}{\nearrow} & \downarrow p \\ U & \hookrightarrow & X \end{array}$$

This means that for each point $x \in U$, there is an open neighbourhood of x, U_x, in which we have an element $s_x \in P(U_x)$, such that $t(x) = \mathsf{germ}_x s_x$. Since t is continuous and $\dot{s}_x(U_x)$ is an open set in Λ_p, there is an open neigbourhood of x, $V_x \subset U_x$, which is mapped by t into $\dot{s}_x(U_x)$, which means $t(V_x) \subset \dot{s}_x(U_x)$. Hence, we have that $t = \dot{s}_x$ on V_x.

This way, we have got a covering of the open set U with the open sets V_x and an element $s_x|_{V_x}$ in each $P(V_x)$. Also, for each pair $x, y \in U$, $s_x|_{V_x\cap V_y} = s_y|_{V_x\cap V_y}$, because $\dot{s}_x$ and $\dot{s}_y$ agree with t on $V_x \cap V_y$ and η is injective. Hence, in the diagram below, with $p_1(\{r_x\}_{x\in U}) = r_x|_{V_x\cap V_y}$ and $p_2(\{r_x\}_{x\in U}) = r_y|_{V_x\cap V_y}$, we get that $p_1(\{s_x\}_{x\in U}) = p_2(\{s_x\}_{x\in U})$.

$$\prod_{x\in U} P(V_x) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{x,y\in U} P(V_x \cap V_y).$$

Therefore, since P is a sheaf, there is the equaliser $P(U) \to \prod_{x \in U} P(V_x)$, so there is an element $s \in P(U)$, such that $s|_{V_x} = s_x$. So, we get that $\forall x \in U$, $t(x) = \mathsf{germ}_x s_x = \mathsf{germ}_x s = \dot{s}(x)$, which means that $t = \dot{s}$ and t is at the image of η_U. As a result, η_U is a bijection, thus η is a natural isomorphism. ☐

Theorem 2.3.10. *For a presheaf P, the sheaf $\Gamma\Lambda_P$, along with the transformation $\eta\colon P \to \Gamma\Lambda_P$, is the universal object from P to sheaves. In other words, for any morphism in the category of presheaves, $\theta\colon P \to C$, with C a sheaf, there is a unique $\sigma\colon \Gamma\Lambda_P \to C$, which makes the diagram below commute:*

$$\begin{array}{ccc} P & \xrightarrow{\eta} & \Gamma\Lambda_P \\ & \searrow^{\theta} & \downarrow{\scriptstyle \exists!\sigma} \\ & & C. \end{array}$$

Proof. We have seen that η is a natural transformation, so the diagram below commutes:

$$\begin{array}{ccc} P & \xrightarrow{\eta} & \Gamma\Lambda_P \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\Gamma\Lambda_\theta} \\ C & \xrightarrow{\eta} & \Gamma\Lambda_C \end{array}$$

Also, as we have proved in Theorem 2.3.9, $\eta\colon C \to \Gamma\Lambda_C$ is an isomorphism, thus it has an inverse η^{-1}. Hence, if we define $\sigma = \eta^{-1} \circ \Gamma\Lambda_\theta$, then $\sigma \circ \eta = \theta$, as desired. It suffices to prove that this σ is unique.

Consider another map $\tau\colon \Gamma\Lambda_P \to C$ with $\tau \circ \eta = \theta$. We take an open set U of X and a cross-section $h \in \Gamma\Lambda_P(U)$. Then, for some $x \in U$, $h(x) \in \Lambda_P(U)$ and $p \circ h(x) = x$, so there is a neighbourhood of x, V_x and an element $s_x \in P(V_x)$, such that $h(x) = \mathsf{germ}_x(s_x) = \dot{s}_x(x)$. From the continuity of h and $\dot{s}_x$, we can choose such a small V_x that $h|_{V_x} = \dot{s}_x = \eta_{V_x}(s_x)$.

As a result, we get $\sigma(h)|_{V_x} = \sigma(h|_{V_x}) = \sigma \circ \eta(s_x) = \theta(s_x) = \tau \circ \eta(s_x) = \tau(h|_{V_x}) = \tau(h)|_{V_x}$. The sets V_x are a cover of the set U, so since C is a sheaf and $\sigma(h), \tau(h) \in C(U)$, we get that $\sigma(h) = \tau(h)$. Since this is for an arbitrary h, we get that $\sigma = \tau$, so σ is unique. ☐

For a given presheaf P, we can consider an "approximation" of P by a sheaf, which is called the sheafification of P and is achieved by applying the natural transformation η.

Corollary 2.3.11 (Sheafification)**.** *For a topological space X, the functor $\Gamma\Lambda\colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$ is called the* ***sheafification functor*** *and it is the left adjoint of the inclusion functor $\mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$.*

Proof. Theorem 2.3.10 proves that the natural transformation $\eta\colon P \to \Gamma\Lambda_P$ is the unit of this adjunction, thus the two functors are adjoint by universal morphisms. As a result, we say that the category of sheaves on X is reflective in the category of presheaves on X. ☐

The procedure of sheafification has proven to be essential for sheaf theory. In the next chapter, in Theorem 3.3.1, we give a generalised way to do it.

2.3.2 Étale Spaces

Definition 2.3.12 (Étale bundle)**.** For topological spaces E and X, a bundle over X, $p\colon E \to X$ is called **étale**, when p is a local homeomorphism in the sense that for each $e \in E$, there is an open set $V \subset E$ with $e \in V$, such that $p(V)$ is open in X and $p|_V\colon V \to p(V)$ is a homeomorphism.

Proposition 2.3.13. *For topological spaces, E and X, if the bundle $p\colon E \to X$ is étale, then p and any cross-sections of p are open maps. Moreover, there is at least one cross-section $s_e\colon U \to E$ through every point $e \in E$ and the images $s_e(U)$ of all these sections form a base for the topology of E. If s and t are two cross-sections, the set $W = \{x \mid s(x) = t(x)\}$ of points where s and t are both defined and have the same value, is open in X.*

The proposition is a consequence of Remark 2.3.7.

Proposition 2.3.14. *For a given bundle on a topological space X, $p\colon Y \to X$, we can construct an étale bundle of the form $\Lambda\Gamma_Y \to X$.*

Remark 2.3.15. Each point of $\Lambda\Gamma_Y$ is of the form $\mathsf{germ}_x s$, where x is a point of $U \subset X$ and $s\colon U \to Y$ is a cross-section of the bundle. Then, we can define the function

$$\epsilon_Y\colon \Lambda\Gamma_Y \to Y, \qquad \mathsf{germ}_x s \mapsto \epsilon_Y(\mathsf{germ}_x s) = s(x).$$

ϵ_Y is well defined, since if $t\colon V \to Y$ is another cross-section with $\mathsf{germ}_x t = \mathsf{germ}_x s$, then there is a neighbourhood of x, $W \subset U \cap V$, such that $s|_W = t|_W$, which gives us that $s(x) = t(x)$.

ϵ_Y is continuous, as well. We will calculate $\epsilon_Y^{-1}(V)$ for an open set $V \subset Y$. If $s(x) \in V$, for a point $x \in X$ and a cross-section $s\colon U \to X$, then $p \circ s(x) \in p(V) \Rightarrow x \in p(V)$. This means that $\epsilon_Y^{-1}(V) = \{\mathsf{germ}_x s\colon x \in p(V), s\colon U \to Y\}$. However, this set is equal with

$$\bigcup_{x\in p(V)} \; \bigcup_{\substack{U\in\mathcal{O}(X),\\ x\in U}} \; \bigcup_{s\in\Gamma_Y(U)} \dot{s}(U),$$

which is a union of open sets, thus an open set in $\Lambda\Gamma_Y$.

Furthermore, ϵ is natural in Y, so it is a natural transformation. This is displayed in the following diagrams.

$$\begin{array}{ccc} \Lambda\Gamma_Y & \xrightarrow{\epsilon_Y} & Y \\ \downarrow{\scriptstyle \Lambda\Gamma f} & & \downarrow{\scriptstyle f} \\ \Lambda\Gamma_{Y'} & \xrightarrow{e_{Y'}} & Y' \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{germ}_x s & \xmapsto{\epsilon_Y} & s(x) \\ \downarrow{\scriptstyle \Lambda\Gamma f} & & \downarrow{\scriptstyle f} \\ \mathsf{germ}_x f \circ s & \xmapsto{e_{Y'}} & f \circ s(x) \end{array}$$

For the morphism $\Lambda\Gamma f$, we used that for a cross-section $s\colon U \to Y$, we have that $p \circ s = i$ ($i\colon U \hookrightarrow X$), and for the map of bundles $f\colon Y \to Y'$, we have that $p' \circ f = p$. So, the map $f \circ s\colon U \to Y'$ is a cross-section of the bundle $p'\colon Y' \to X$, since $p' \circ f \circ s = p \circ s = i$, as wanted. This means that $\Lambda\Gamma f(\mathsf{germ}_x s) = \mathsf{germ}_x f \circ s$.

Theorem 2.3.16. *If the bundle of topological spaces $p\colon Y \to X$ is étale, then the natural transformation ϵ with $\epsilon_Y(\mathsf{germ}_x s) = s(x)$ is a natural isomorphism, $\Lambda\Gamma_Y \cong Y$.*

Proof. We will prove that ϵ_Y is an isomorphism by constructing an inverse, $\epsilon'_Y \colon Y \to \Lambda\Gamma_Y$. Considering a point $y \in Y$, we have that $p(y) = x \in X$. Since p is étale, there is a neighbourhood V of y, such that $p|_V \colon V \to p(V)$ is a homeomorphism. Thus, for the open set $U = p(V)$, there is a cross-section $s \colon U \to Y$ with $s(x) = y$. Then, we can define $\epsilon'_Y(y) = \mathsf{germ}_x s \in \Lambda\Gamma_Y$.

$\epsilon'_Y(y)$ is well-defined, since if $t \colon U' \to Y$ is another cross-section of the bundle with $t(x) = y$, then $s(x) = t(x)$, so from continuity of s and t there is a neighbourhood of x, $W \subset U \cap U'$, such that $s|_W = t|_W$, which means that $\mathsf{germ}_x s = \mathsf{germ}_x t$.

ϵ'_Y is continuous, as well. For a base open set $\dot{s}(U) = \{\mathsf{germ}_x s \colon x \in U, s \colon U \to Y\} \subset \Lambda\Gamma_Y$, we have

$$(\epsilon'_Y)^{-1}(\dot{s}(U)) = \{y \colon s(x) = y, x \in U\} = s(U).$$

$s(U)$ is an open set, because of the properties of étale spaces. Hence, we have the desired continuity.

Finally,

$$\epsilon_Y(\epsilon'_Y(y)) = \epsilon_Y(\mathsf{germ}_x s) = s(x) = y,$$

and

$$\epsilon'_Y(\epsilon_Y(\mathsf{germ}_x s)) = \epsilon'_Y(s(x)) = \mathsf{germ}_x s.$$

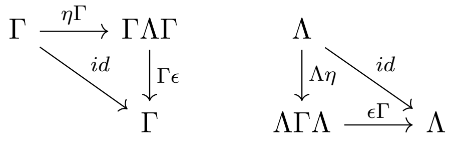
As a result ϵ_Y is a bijection and ϵ a natural isomorphism. □

Theorem 2.3.17. *For any topological space X, the following functors form an adjoint pair:*

$$\Lambda \colon \mathbf{PSh}(X) \rightleftarrows \mathbf{Bund}(X) \colon \Gamma.$$

Proof. We have already constructed the natural transformation $\eta_P \colon P \to \Gamma\Lambda P$ (in Theorems 2.3.9 and 2.3.10), which maps each $s \in PU$ to the corresponding cross-section $\dot{s} \colon U \to \Lambda(P)$, and the natural transformation $\epsilon_Y \colon \Lambda\Gamma_Y \to Y$, which maps each $\mathsf{germ}_x s$ to $s(x)$. We will show that these are the unit and counit of the adjunction, respectively.

For that purpose, we prove that the triangular identities hold:



We go pointwise, so for the first identity, we consider a bundle $p \colon Y \to X$ and a cross-section $s \in \Gamma_p U$. Then $s \overset{\eta\Gamma}{\longmapsto} \dot{s} \overset{\Gamma\epsilon}{\longmapsto} s$, since for any $x \in U$, $\epsilon(\dot{s}(x)) = \epsilon(\mathsf{germ}_x s) = s(x)$. Similarly, for the second identity, we consider $\mathsf{germ}_x s \in \Lambda(P)$ and then $\mathsf{germ}_x s \overset{\Lambda\eta}{\longmapsto} \mathsf{germ}_x \dot{s} \overset{\epsilon\Lambda}{\longmapsto} \dot{s}(x) = \mathsf{germ}_x s$. □

Corollary 2.3.18. *The adjoint pair (Γ, Λ) restrict to an equivalence of categories $\mathbf{Sh}(X) \simeq \mathbf{Etale}(X)$.*

Proof. The natural transformations $\eta \colon I_{\mathbf{PSh}(X)} \to \Gamma\Lambda$ and $\epsilon \colon \Lambda\Gamma \to I_{\mathbf{Bund}(X)}$ are isomorphisms as stated in Theorems 2.3.9 and 2.3.16, respectively. □

Corollary 2.3.19. *We can describe a morphism between two sheaves F and G on X, $h \colon F \to G$, in three equivalent ways:*

1. *as a natural transformation of functors $h \colon F \to G$*

2. *as a continuous map of the associated bundles over* X, $h\colon \Lambda_F \to \Lambda_G$

3. *as a family of functions* $h_x\colon F_x \to G_x$, *on the respective fibers over each* $x \in X$, *such that for each open set* $U \subset X$ *and each* $s \in F(U)$, *the function* $U \to \Lambda_G$ *with* $x \mapsto h_x(\dot{s}(x))$ *is continuous.*

Proof. The equivalence of (a) and (b) is proven in Corollary 2.3.18.

In order to obtain (c) from (a), consider $h_U\colon F(U) \to G(U)$ natural in U. Then, for each $x \in X$ we can determine a unique $h_x\colon F_x \to G_x$ such that the diagram commutes (which means we have $h_x(\mathsf{germ}_x s) = \mathsf{germ}_x h_U(s)$).

$$\begin{array}{ccc} F(U) & \xrightarrow{h_U} & G(U) \\ {\scriptstyle \mathsf{germ}_x}\downarrow & & \downarrow{\scriptstyle \mathsf{germ}_x} \\ F_x & \xrightarrow{h_x} & G_x \end{array}$$

Since $\dot{s}(x) = \mathsf{germ}_x s$ for $x \in U$, from the diagram we have

$$h_U \circ \dot{s}(x) = \mathsf{germ}_x(h_U(s)) = h_x(\mathsf{germ}_x s) = h_x(\dot{s}(x)),$$

thus the function $x \mapsto h_x(\dot{s}(x))$ is actually $h_U \circ \dot{s}$, which is continuous.

Conversely, considering a family $h_x\colon F_x \to G_x$ with this continuity condition, we can construct h_U like above or equivalently, by taking the disjoint union over the fibers, the map of bundles $h\colon \Lambda_F \to \Lambda_G$. □

2.4 Sheaves with Algebraic Structure

In algebraic geometry and algebraic topology, many times we need to take the homology and cohomology groups locally, which are essentially sheaves of groups. This was one of the main motivations to consider the concept of sheaf. Thus, we need a way to define systematically sheaves of groups, rings or any other algebraic structure, starting from the sheaves of sets we have already defined. We will do this by diagrams. We show the case of an abelian group, but any algebraic structure can be recreated in the same way.

An abelian group is a set A with a binary operation (addition), a unary operation (inverse) and a nullary operation (zero; nullary because it is a function defined on a single object-the terminal object). Thus we have:

$$A \times A \xrightarrow{a} A, \quad (x, y) \mapsto x + y,$$

$$A \xrightarrow{v} A, \quad x \mapsto -x,$$

$$1 \xrightarrow{u} A, \quad \{*\} \mapsto 0.$$

These operations satisfy certain identities: associative and commutative laws for a, v a left inverse for a and u a left zero. Below we can see the respective commuting diagrams which should be satisfied:

Associativity:

$$\begin{array}{ccc} A \times A \times A & \xrightarrow{\mathsf{id} \times a} & A \times A \\ \big\downarrow{\scriptstyle a \times \mathsf{id}} & & \big\downarrow{\scriptstyle a} \\ A \times A & \xrightarrow{\quad a \quad} & A \end{array}$$

where $\mathsf{id}\colon A \to A$ is the identity function.

Commutativity:

$$\begin{array}{ccc} A \times A & & \\ \big\downarrow{\scriptstyle \mathsf{rev}} & \searrow{\scriptstyle a} & \\ A \times A & \xrightarrow{a} & A \end{array}$$

where $\mathsf{rev}\colon A \times A \to A \times A$ is the function with $(x, y) \mapsto (y, x)$.

Left inverse:

$$\begin{array}{ccccc} A & \xrightarrow{\mathsf{diag}} & A \times A & \xrightarrow{v \times \mathsf{id}} & A \times A \\ \big\downarrow & & & & \big\downarrow{\scriptstyle a} \\ \{*\} & & \xrightarrow{\qquad u \qquad} & & A \end{array}$$

where $\mathsf{diag}\colon A \to A \times A$ is the function with $x \mapsto (x, x)$ and the function $A \to \{*\}$ is the unique morphism to the terminal object (the unique function to the one-element set).

Left zero:

$$\begin{array}{ccc} \{*\} \times A & \xrightarrow{u \times \mathsf{id}} & A \times A \\ \big\downarrow{\scriptstyle \mathsf{pr}_2} & \swarrow{\scriptstyle a} & \\ A & & \end{array}$$

where $\mathsf{pr}_2\colon \{*\} \times A \to A$ is the function with $(*, x) \mapsto x$.

Definition 2.4.1 (Abelian group object)**.** In any category $\mathcal{C}$ with finite limits (so a terminal object as well) we can define an **abelian group object** of $\mathcal{C}$ to be an object A, together with three morphisms a, v, u as defined above, which make the aforementioned four diagrams commute.

The abelian group objects of $\mathcal{C}$ form the objects of the category $\mathbf{Ab}(\mathcal{C})$ with morphisms all the morphisms $f\colon A \to A'$ that commute with a, v and u.

It is easy to observe that $\mathbf{Ab} = \mathbf{Ab}(\mathbf{Set})$. We can formulate similar definitions for rings, modules, etc. and any algebraic structure defined by one or more n-ary operations which satisfy specific identities.

Now we can apply this definition to define sheaves of abelian groups:

Definition 2.4.2 (Presheaf of abelian groups)**.** A **presheaf of abelian groups**, P, on a topological space X is an abelian group object in $\mathbf{PSh}(X)$ (or equivalently an object of the category $\mathbf{Ab}(\mathbf{PSh}(X))$).

We can see that since the product in the category $\mathbf{PSh}(X)$ is the pointwise product in **Set**, the operation of addition a for such a P is restricted to $a_U\colon P(U) \times P(U) \to P(U)$ natural to $U \in \mathcal{O}(X)$, thus inheriting the properties of addition. The same holds for the other operations, therefore $P(U)$ is an abelian group. We can conclude that we can describe a presheaf of abelian groups on X as a functor $P\colon \mathcal{O}(X)^{\mathsf{op}} \to \mathbf{Ab}$.

Similarly, we can define a **sheaf of abelian groups** on X as an abelian object of $\mathbf{Sh}(X)$ or equivalently a functor $F\colon \mathcal{O}(X)^{\mathsf{op}} \to \mathbf{Ab}$ such that the composite with the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is a sheaf of sets.

In a similar way, we can define a bundle of abelian groups over the space X: simply it is an abelian group object in the category $\mathbf{Bund}(X)$. In this category the product of the bundle $p\colon Y \to X$ is the pullback of p with itself in $\mathbf{Top}$:

$$\begin{array}{ccc} Y \times_X Y & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{\ p\ } & X \end{array}$$

Considering the definition of pullback in $\mathbf{Top}$, we conclude that $Y \times_X Y$ is the subspace of $Y \times Y$ with elements all the pairs (y, y') of points of Y that lie in the same fiber of p. As a result, addition in the group object Y is defined by addition in each fiber $p^{-1}(x)$. Thus we have an equivalent definition of bundle of abelian groups: it is a bundle of spaces $p\colon Y \to X$, such that each fiber $p^{-1}(x)$ is an abelian group in a way such that the generalised operations $a\colon Y \times_X Y \to Y$ and $v\colon Y \to Y$ are continuous maps and bundles over X. Specifically, for $Y \times_X Y$, we must have the following commutative diagram:

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{\ a\ } & Y \\ & \searrow^{p'} & \big\downarrow{\scriptstyle p} \\ & & X \end{array}$$

where $p'\colon Y \times_X Y \to X$ is the map induced by the pullback square.

We have the same for étale spaces, as well, since the product of two étale spaces over X is étale over X. This is immediate from the fact that the pullback of two local homeomorphisms is a local homeomorphism (Lemma 2.5.3).

For categories $\mathcal{C}$ and $\mathcal{D}$, if a functor $\Lambda\colon \mathcal{C} \to \mathcal{D}$ preserves finite products, then it is easy to see that it maps any abelian group object in $\mathcal{C}$ to another abelian group object in $\mathcal{D}$. Thus, it induces a functor $\mathbf{Ab}(\mathcal{C}) \to \mathbf{Ab}(\mathcal{D})$.

Hence, in order to transfer the above results on sheaves of sets to sheaves of abelian groups, it suffices to show that the functors $\Gamma\colon \mathbf{Bund}(X) \to \mathbf{PSh}(X)$ and $\Lambda\colon \mathbf{PSh}(X) \to \mathbf{Bund}(X)$ preserve finite limits. Γ preserves finite limits, since for bundles $p\colon Y \to X$ and $p'\colon Y' \to X$, a cross-section of the product bundle $Y \times_X Y'$ is just a pair of cross-sections s and s', one of Y and one of Y' respectively (it is a cross-section, because $Y \times_X Y'$ is pullback, so the compositions of s with p and of s' with p' are inclusions). Equivalently, we could say that Γ is a right adjoint, so it preserves all limits.

In the case of Λ, each fiber of Λ_P at x, P_x, is a colimit over the subcategory of $\mathcal{O}(X)^{\mathsf{op}}$ with all open sets U that contain x, as we have seen in Remark 2.3.5. In this subcategory, if we have two sets U and V, then their intersection is also in the subcategory, which means that it is a "filtered" subcategory, which makes fibers filtered colimits. However, filtered colimits commute with finite limits, hence Λ also preserves finite products (for more details, see [1])

As a result, we get the same results as in Theorem 2.3.17 and Corollaries 2.3.18 and 2.3.11 for abelian groups. Thus we get a pair of adjoint functors:

$$\Lambda\colon \mathbf{Ab}(\mathbf{PSh}(X)) \rightleftarrows \mathbf{Ab}(\mathbf{Bund}(X)\colon \Gamma)$$

which restrict to an equivalence of the categories $\mathbf{Ab}(\mathbf{Sh}(X)) \simeq \mathbf{Ab}(\mathbf{Etale}(X))$. Also, the inclusion $\mathbf{Ab}(\mathbf{Sh}(X)) \hookrightarrow \mathbf{Ab}(\mathbf{Set}^{\mathcal{O}(X)^{\mathsf{op}}})$ is reflective, which means that it has a left adjoint, the sheafification functor of abelian groups. The same results also hold for rings, modules, etc.

2.5 Direct and Inverse Image

Definition 2.5.1 (Direct image)**.** If $f\colon X \to Y$ is a continuous map of topological spaces, then for each sheaf F on X, we can define a new sheaf f_*F, called **direct image**, such that for any open V in Y, $f_*F(V) = F(f^{-1}(V))$, or diagrammatically it is the composition

$$\mathcal{O}(Y)^{op} \xrightarrow{f^{-1}} \mathcal{O}(X)^{op} \xrightarrow{F} \mathbf{Set}.$$

Then, $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ is a functor. Also, $(f \circ g)_* = f_* \circ g_*$, so by defining $\mathsf{Sh}(f) = f_*$, we have made Sh a functor on the category of topological spaces.

Remark 2.5.2. If $f\colon X \to Y$ is a homeomorphism, then f_* is an isomorphism of the categories $\mathbf{Sh}(X)$ and $\mathbf{Sh}(Y)$.

For a given continuous map $f\colon X \to Y$, we can also define a functor between sheaves in the opposite direction; $f^*\colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$. We will define this using the equivalence of sheaves with étale spaces. Considering a bundle over Y, $p\colon E \to Y$, we can define a bundle over X, $f^*p\colon f^*E \to X$ as the pullback of p along f.

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \big\downarrow{\scriptstyle f^*p} & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

f^* is a functor and furthermore we have the following result:

Lemma 2.5.3. *If $f\colon X \to Y$ is a continuous map between topological spaces and $p\colon E \to Y$ is an étale space over Y, then $f^*E \to X$ is étale over X.*

Proof. The pullback f^*E is the space $\{\langle x, e\rangle \colon x \in X, e \in E, f(x) = p(e)\}$. Taking a point $\langle x, e\rangle$, we have that since p is étale, there is an open neighbourhood U of e in E, which is mapped homeomorphically by p onto an open set $p(U)$ in X. Then, from continuity of f, we get that $f^{-1}(p(U)) \times U$ is an open set in $X \times E$ containing $\langle x, e\rangle$, thus $(f^{-1}(p(U)) \times U) \cap f^*E$ is an open neighbourhood of $\langle x, e\rangle$ in the pullback and it is mapped homeomorphically onto $f^{-1}(p(U))$ in X. Therefore, $f^*p\colon f^*E \to X$ is étale. □

Definition 2.5.4 (Inverse image)**.** Now, using the equivalence of categories in Corollary 2.3.18 and Lemma 2.5.3, we can define $f^*\colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ as the composition of:

$$\mathbf{Sh}(Y) \xrightarrow{\Lambda} \mathbf{Etale}(Y) \xrightarrow{f^*} \mathbf{Etale}(X) \xrightarrow{\Gamma} \mathbf{Sh}(X).$$

For a sheaf G on Y, we call $f^*(G) \in \mathbf{Sh}(X)$‘ the **inverse image** of G.

Theorem 2.5.5. *Let $f\colon X \to Y$ be a continuous map, then the inverse image functor $f^*\colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ is left adjoint to the direct image functor $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.*

$$f^*\colon \mathbf{Sh}(Y) \rightleftarrows \mathbf{Sh}(X)\colon f_*$$

For proof, see [2].

2.6 The category of sheaves is a topos

In this section, we recall the notion of elementary topos and we show that the category of sheaves is a topos.

Definition 2.6.1 (Elementary topos)**.** An **elementary topos** is a category which

- has finite limits,
- is cartesian closed (has exponentials), and
- has a subobject classifier.

In this section, we will show that the category $\mathbf{Sh}(X)$ of the sheaves of sets over a space X is a topos. We start by proving the existence of all finite limits.

Proposition 2.6.2. *The category $\mathbf{Sh}(X)$ of the sheaves of sets over a space X has all finite limits and the inclusion of sheaves in the category of presheaves preserves these limits.*

See proof in [2], II.8.

From the previous proposition, we get a useful corollary for later.

Corollary 2.6.3. *Consider a sheaf F and a subobject of F represented by the monic $m\colon S \rightarrowtail F$. Then S is isomorphic to a subsheaf of F.*

Proof. Since m is a monic, we have the pullback square in $\mathbf{Sh}(X)$:

$$\begin{array}{ccc} S & \xrightarrow{\mathsf{id}} & S \\ \big\downarrow{\scriptstyle \mathsf{id}} & & \big\downarrow{\scriptstyle m} \\ S & \xrightarrow{m} & F \end{array}$$

By Proposition 2.6.2, this is also a pullback in the category of presheaves on X, $\mathbf{PSh}(X)$. In this category, pullbacks are taken pointwise, which means that each m_U for every open $U \subset X$ is monic. Therefore, $S(U)$ is isomorphic to a subset $S'(U)$ of $F(U)$ and thus S is isomorphic to the subfunctor S' of F. Finally, S' is a sheaf, because it is isomorphic to the sheaf S. $\square$

The second property we need is the existence of all exponentials.

Proposition 2.6.4. *Consider F a sheaf and P a presheaf on a topological space X. Then the exponential presheaf F^P is a sheaf.*

Proof. For the definition of the exponential, we see F and P as functors $\mathcal{O}(X)^{\mathsf{op}} \to \mathbf{Set}$. Then, the exponent is the functor $F^P\colon \mathcal{O}(X)^{\mathsf{op}} \to \mathbf{Set}$ such that for an open $U \subset X$:

$$F^P(U) = \mathsf{Hom}(\mathbf{y}(U) \times P, F)$$

where Hom denotes the hom-set in the functor category $\mathbf{Set}^{\mathcal{O}(X)^{\mathsf{op}}} = \mathbf{PSh}(X)$, in other words the natural transformations, and $\mathbf{y}(U) = \mathsf{Hom}_{\mathcal{O}(X)}(-, U)$ is the Yoneda embedding, which in the respected category means that $\mathbf{y}(U)(V)$ is $\{*\}$ if $V \subset U$, otherwise it is empty. As a result, the natural transformations in $F^P(U)$ are defined only for open sets $V \subset U$, so we get

$$F^P(U) \cong \mathsf{Hom}(P|_U, F|_U)$$

where now the hom-set is in the category of presheaves on U and $P|_U$ and $F|_U$ are the functors P and F restricted to $\mathcal{O}(U)^{\mathsf{op}}$.

It is easy to see that $F^P(U)$ is a functor of U: for every natural transformation $a\colon P|_U \to F|_U$, if $V \subset U$, then we get the restriction $a|_V\colon P|_V \to F|_V$, which means that $F^P(U) \subset F^P(V)$. Hence, F^P is the presheaf on $\mathcal{O}(X)^{\mathsf{op}}$ which maps any open U to the maps $P|_U \to F|_U$ of presheaves on U.

Considering a covering $\bigcup_i U_i = U$ and natural transformations $\tau_i\colon P|_{U_i} \to F|_{U_i}$ for all i, we can construct the transformation $\tau\colon P|_U \to F|_U$ by collation of the respective τ_i. An easy way to see this is that a natural transformation is computed pointwise, so for open $V \subset U$, $\tau_V(P|_U(V))$ is the set resulting from the collation of $\tau_{i,V}(P|_{U_i}(V)) \in F|_{U_i}(V)$, which is possible, since F is a sheaf. $\square$

Next, we need to define a subobject classifier Ω for $\mathbf{Sh}(X)$. So, we define the presheaf on X, Ω, by taking $\Omega(U)$ to be the set of all open subsets of U for every open $U \subset X$:

$$\Omega(U) = \{W \mid W \subset U, W \mathsf{openin} X\}.$$

This is clearly a functor, because if $V \subset U$, then we get the map $\Omega(U) \to \Omega(V)$ with $W \mapsto W \cap V$.

Proposition 2.6.5. *The presheaf Ω on the topological space X is a sheaf on X and a subobject classifier in the category $\mathbf{Sh}(X)$.*

Proof. We consider an arbitrary open set $U \subset X$ and a covering of it $\bigcup_i U_i = U$. Then, we take for every i, elements of the sets $\Omega(U_i)$, which means open sets $V_i \subset U_i$ which agree in the intersection, so for all i, j, $V_i \cap U_j = V_j \cap U_i \subset U_i \cap U_j$ ($V_i \cap U_j$ is the mapping of V_i through the morphism $U_i \to U_i \cap U_j$ as defined above). Then, we can get a unique open set $V \in \Omega(U)$, such that for any i the restriction $V \cap U_i = V_i$. This is the union of V_i's, because $V \cap U_i = \bigcup_j V_j \cap U_i = \bigcup_j (V_j \cap U_i) = \bigcup_j (V_i \cap U_j) = V_i \cap \bigcup_j U_j = V_i \cap U = V_i$. If there is another V' with $V' \cap U_i = V_i$, then $V \cap V' = \bigcup_i V_i \cap V' = \bigcup_i (V_i \cap V') \subset \bigcup_i (U_i \cap V') = \bigcup_i V_i = V$, thus $V \subset V'$, which means that V is an element of the equaliser. Therefore, $\Omega(U)$ is the equaliser and Ω is a sheaf.

Next let $S \subset F$ be a subobject of the sheaf F. From Corollary 2.6.3, we can assume that S is a subsheaf of F. In other words, for each open $U \subset X$, $S(U)$ is a subset of $F(U)$.

We define as the suggested characteristic natural transformation $\phi\colon F \to \Omega$ with the functions $\phi_U\colon F(U) \to \Omega(U)$ for each open $U \subset X$, which map each $x \in F(U)$ to the union W of all the open subsets $W_i \subset U$ with $x|_{W_i} \in S(W_i)$. Then, $x|_W \in S(W)$, since S is a sheaf, and ϕ_U is natural in U. To see the latter, we present the commutative diagram below. For an open $V \subset U$:

$$\begin{array}{ccc} F(U) & \xrightarrow{\phi_U} & \Omega(U) \\ \downarrow & & \downarrow \\ F(V) & \xrightarrow{\phi_V} & \Omega(V) \end{array} \qquad \begin{array}{ccc} x & \xmapsto{\phi_U} & W \\ \downarrow & & \downarrow \\ x|_V & \xmapsto{\phi_V} & W \cap V \end{array}$$

Now, we define the map $\mathsf{true}\colon \mathbb{1} \to \Omega$, where $\mathbb{1}$ is the terminal object of $\mathbf{Sh}(X)$, as the map that for each U sends the point $\{*\}$ to U. We then consider the pullback of true along ϕ and we take it pointwise, as well:

$$\begin{array}{ccc} P & \dashrightarrow & \mathbb{1} \\ \downarrow & & \downarrow \mathsf{true} \\ F & \xrightarrow{\phi} & \Omega \end{array} \qquad \begin{array}{ccc} P(U) & \dashrightarrow & \mathbb{1} \\ \downarrow & & \downarrow \mathsf{true} \\ F(U) & \xrightarrow{\phi} & \Omega(U) \end{array}$$

As a result, $P(U)$ is the subset of $F(U)$ with all those $x \in F(U)$, such that $\phi_U(x) = U$. But these elements are exactly the elements of $S(U)$ (if $x \in S(U)$ then $x|_W \in S(W)$ for any open $W \subset U$, so $\phi_U(x) = U$ and if $\phi_U(x) = U$, we showed above that $x = x|_U \in S(U)$). We conclude that the pullback P is exactly the subsheaf S.

Furthermore, if S is a pullback along any other $\psi\colon F \to \Omega$, then $\psi|_U(x) = U$ if and only if $x \in S(U)$. Also, since ψ is a natural transformation, for $V \subset U$ we get the commutative diagram:

$$\begin{array}{ccc} F(U) & \xrightarrow{\psi_U} & \Omega(U) \\ \downarrow & & \downarrow \\ F(V) & \xrightarrow{\psi_V} & \Omega(V) \end{array} \qquad \begin{array}{ccc} x & \xmapsto{\psi_U} & \psi(x) \\ \downarrow & & \downarrow \\ x|_V & \xmapsto{\psi_V} & \psi(x) \cap V \end{array}$$

Hence, combining these two properties, we get $\psi(x) \cap V = V$ if and only if $x|_V \in S(V)$. This shows immediately that $\bigcup_{x \in S(V)} V \subset \psi(x)$. Also, assuming there is an open set $W \subset \psi(x)$ with $W \not\subset \bigcup_{x \in S(V)} V$, we have that $\psi(x) \cap W = W$, thus $x|_W \in S(W)$, which means that $W \subset \bigcup_{x \in S(V)} V$. Therefore, $\psi(x) = \bigcup_{x \in S(V)} V$, which means that $\psi = \phi$, so ϕ is unique.

Concluding, Ω is a sheaf and the map $\mathsf{true}\colon \mathbb{1} \to \Omega$ is a subobject classifier. □

Finally, we have the conclusion we desired:

Corollary 2.6.6. *The category of sheaves over a topological space X, $\mathbf{Sh}(X)$, is an elementary topos.*

Proof. We have shown that $\mathbf{Sh}(X)$ has all finite limits (Proposition 2.6.2), exponentials (Proposition 2.6.4) and a subobject classifier (Proposition 2.6.5). It suffices to show that it has all colimits, too.

The sheafification functor $\Gamma\Lambda\colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$ is left adjoint to the inclusion functor from Corollary 2.3.11, thus it preserves colimits. Also, for a sheaf F it is the isomorphic to the identity functor $\Gamma\Lambda(F) \cong F$. So, if we have some finite diagram of sheaves A, we can take their colimit in the functor category (where all colimits exist). Then, since A is a diagram of sheaves, $\Gamma\Lambda(A) \cong A$,

$$\varinjlim_{\mathbf{PSh}(X)} (A) \cong \varinjlim_{\mathbf{PSh}(X)} (\Gamma\Lambda(A)) \cong \Gamma\Lambda(\varinjlim_{\mathbf{PSh}(X)} (A))$$

(last isomorphism comes from the fact that $\Gamma\Lambda$ preserves colimits. Hence, the colimit of A is a sheaf, in other words there is the colimit of any diagram of sheaves.

Therefore, we get that $\mathbf{Sh}(X)$ is an elementary topos. $\square$

Chapter 3

Grothendieck Topology and Sheaves on Sites

The main idea in this chapter is to generalise the notion of sheaves from the category of open spaces of a topological space to any category that this notion may be useful. For this, we need an alternative notion of covering of an object, which we call Grothendieck topology. A site then, a category with its Grothendieck topology, is the space where we define the sheaves on.

3.1 Grothendieck Topology

We start by giving a more general definition of sieve.

Definition 3.1.1 (Sieve)**.** Given an object K in the category $\mathcal{C}$, a **sieve** on K is a set S of morphisms with codomain K such that if $f \in S$ and for another morphism h, $f \circ h$ is defined, then $f \circ h \in S$.

We can think of a sieve as "the set of paths that are allowed to lead to K". Then, if there is a path to go somewhere and from there, there is an allowed path to K, then you are allowed to do so.

Proposition 3.1.2. *Given an object K in the category $\mathcal{C}$, a sieve on K is equivalent with a subfunctor of the Yoneda embedding $\mathbf{y}(K) = \mathsf{Hom}_{\mathcal{C}}(-, K)$.*

Proof. Given a sieve S on K, we can define

$$Q(A) = \{f \mid f\colon A \to K, f \in S\},$$

which is a subset of $\mathsf{Hom}_{\mathcal{C}}(A, K)$. However, it is easy to see that Q is a functor $\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$. Therefore, Q is a subfunctor of $\mathbf{y}(K)$.

Conversely, given a subfunctor Q of $\mathbf{y}(K)$, we can define the set

$$S = \{f \mid \exists A \in \mathsf{Ob}(\mathcal{C}) \;\; f\colon A \to K, f \in Q(A)\}.$$

Then, if for a morphism in S, $f\colon A \to K$ and another morphism $h\colon B \to A$ ($f \circ h$ is defined), we see that $f \circ h = Q(h)(f) \in Q(B)$, which means that $f \circ h \in S$, so S is a sieve. $\square$

We will try to create the notion of a sheaf on a category, step by step, in analogy with the case of topological spaces. We consider a category $\mathcal{C}$ and an object $K \in \mathsf{Ob}(\mathcal{C})$. We can define presheaves on $\mathcal{C}$ as the functors $F \colon \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$.

The next step for the description of sheaves is to define what are the coverings of K. In analogy to sheaves on topological spaces, where a covering is a collection of subsets of the initial set, thus open sets that are mapped to the initial set by morphisms with codomain to the initial set, we set a covering to be a collection of maps with codomain K, so

$$S = \{f_i \colon K_i \to K \mid i \in I\}.$$

Since every open set may have different coverings, we may assume that for each object there are also different coverings, so we denote with $\mathcal{J}(K)$ all the different coverings of K.

Note that for a covering $S = \{f_i \colon K_i \to K \mid i \in I\}$, we can isolate an object K_m and consider a covering of it $S_m = \{g_j \colon K_{mj} \to K_m\}$. Then, the covering $S \setminus \{f_m\} \cup \{f_m \circ g_j \colon K_{mj} \to K\}$ is essentially the same covering as S. Hence, in order to contain all these equivalent coverings, we can consider a covering S to be a sieve.

Now, we can summarise these ideas in the concept of Grothendieck topology, where there are also some properties added.

Definition 3.1.3 (Grothendieck topology)**.** A **Grothendieck topology** on a category $\mathcal{C}$ is a function $\mathcal{J}$ which maps each object K of $\mathcal{C}$ to a collection $\mathcal{J}(K)$ of sieves, such that $\mathcal{J}$ satisfies the properties:

1. The **maximal sieve** $S_K^{\mathsf{max}} = \{f \mid \mathsf{codom}(f) = K\}$ (which is equivalent with the Yoneda embedding $\mathbf{y}(K)$) is in $\mathcal{J}(K)$.

2. (Stability axiom) If $S \in \mathcal{J}(K)$, then for any morphism $h \colon A \to K$, $h^*(S) = \{g \mid \mathsf{codom}(g) = A, h \circ g \in S\} \in \mathcal{J}(A)$.

3. (Transitivity axiom) If $S_1 \in \mathcal{J}(K)$ and S_2 is a sieve on K with $h^*(S_2) \in \mathcal{J}(A)$ for any $h \colon A \to K$ in S_1, then $S_2 \in \mathcal{J}(K)$.

Definition 3.1.4 (Site)**.** We call the pair $\mathbf{C} = (\mathcal{C}, \mathcal{J})$ of a category and a Grothendieck topology on it, a **site**. The elements of the collection $\mathcal{J}(K)$ for an object K are called **covers**.

We give some examples of Grothendieck topologies.

1. The **trivial topology** on the category $\mathcal{C}$ has only one sieve for each object K, which is the maximal sieve S_K^{max}. This is obviously the minimal topology we can have.

2. The **atomic topology** on the category $\mathcal{C}$ is the one with covers all the non-empty sieves ($S \in \mathcal{J}(K) \iff S$ is non-empty). In order the topology to satisfy the stability axiom, we need a condition called the **Ore condition**, which is whenever we have two morphisms with the same codomain, $f \colon A \to K$ and $g \colon B \to K$, then we can complete the Ore square such that it is commutative (a condition weaker than the condition of pullbacks):

$$\begin{array}{ccc} \bullet & \dashrightarrow & A \\ \downarrow & & \downarrow f \\ B & \xrightarrow{g} & K. \end{array}$$

Particularly, we can show that the atomic topology is a Grothendieck topology if and only if the category $\mathcal{C}$ satisfies the Ore condition.

3.2 Sheaves on a Site

For the classical definition of sheaves, we need to find a way to express the union of two open sets. But this is just the fiber product of two maps. This means that if we have the pullback square

$$\begin{array}{ccc} K_i \times_K K_j & \xrightarrow{h_{ij}} & K_j \\ \downarrow{\scriptstyle v_{ij}} & & \downarrow{\scriptstyle f_j} \\ K_i & \xrightarrow{f_i} & K, \end{array}$$

then applying the presheaf $P\colon \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$, we get:

$$\begin{array}{ccc} P(K_i \times_K K_j) & \xleftarrow[P(h_{ij})]{} & P(K_j) \\ {\scriptstyle P(v_{ij})}\uparrow & & {\scriptstyle P(f_j)}\uparrow \\ P(K_i) & \xleftarrow[P(f_i)]{} & P(K). \end{array}$$

In the end, P is a sheaf if for a cover $S \in \mathcal{J}(K)$ the diagram below is an equaliser:

$$P(K) \overset{e}{\dashrightarrow} \prod_{f_i \in S} P(K_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{f_i, f_j \in S} P(K_i \times_K K_j),$$

However, it is possible that the category of interest does not have fiber products. So, we define the sheaves on a site in analogy with the equivalent definition in Proposition 2.2.1, which in this case is proven to be more general.

Definition 3.2.1 (Sheaf on a site)**.** A presheaf P on a site $\mathbf{C}$ is a **sheaf** if and only if for every covering sieve S on K, the inclusion functor $S \hookrightarrow \mathbf{y}(K)$ induces the isomorphism

$$\mathsf{Hom}_{\mathbf{PSh}(\mathcal{C})}(S, P) \cong \mathsf{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathbf{y}(K), P).$$

In more detail, if we have a presheaf $P\colon \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ on the site $\mathbf{C} = (\mathcal{C}, \mathcal{J})$ and a sieve S which is a cover of the object K, we say that a **matching family** for S with elements of P is a function which maps each element $f\colon A \to K$ of S to the element $x_f \in P(A)$, such that $P(g)(x_f) = x_{f \circ g}$ ($f \circ g \in S$ since S is a sieve) for every morphism $g\colon B \to A$. We can see how matching family compares with open sets that agree in their intersection, which is in the definition of sheaves on topological spaces. So, the next thing to introduce for the definition of sheaf is a single element $x \in P(K)$ that can generate all the elements of a matching family, in other words $x_f = P(f)(x)$ for every $f \in S$. We call this an **amalgamation** of this matching family.

Therefore, P is a sheaf for the Grothendieck topology $\mathcal{J}$, if for every matching family of P for every cover of any object in $\mathbf{C}$, there is a unique amalgamation.

We can express this with an equaliser diagram: We say that a presheaf P on $\mathbf{C}$ is a sheaf if and only if for every object K and any cover $S \in \mathcal{J}(K)$ the following diagram is an equaliser.

$$P(K) \overset{e}{\dashrightarrow} \prod_{f\in S} P(\mathsf{dom}(f)) \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{\substack{f\in S,\\ \mathsf{dom}(f)=\mathsf{codom}(g)}} P(\mathsf{dom}(g))$$

Here we have the map $e(x) = \{P(f)(x)\}_{f\in S}$ and the other two maps, which regard a matching family, so we have $p_1(\{x_f\}_{f\in S}) = \{x_{f\circ g}\}_{f,g}$ and $p_2(\{x_f\}_{f\in S}) = \{P(g)(x_f)\}_{f,g}$.

In order to see the equivalence of this definition with the previous one, we can consider a matching family $f \mapsto x_f$ of a cover S of an object K, as a natural transformation from S to P (and conversely). So, if there is a unique amalgamation x, then we can consider it to be the mapping $1_K \mapsto x$, which gives us $f \mapsto P(f)(x)$ for any f with codomain K, which is a unique natural transformation from $\mathbf{y}(K)$ to P.

Definition 3.2.2 (Grothendieck topos)**.** A **Grothendieck topos** is a category which is equivalent to some category $\mathbf{Sh}(\mathbf{C})$ on a site $\mathbf{C} = (\mathcal{C}, \mathcal{J})$.

3.3 Sheafification

Like in the case of sheaves on a topological space, there is a functor $\mathbf{a}\colon \mathbf{PSh}(\mathcal{C}) \to \mathbf{Sh}(\mathbf{C})$, which maps every presheaf on $\mathcal{C}$ to an associated sheaf. This process is called sheafification and we denote the associated sheaf of the presheaf P, as $P^\# = \mathbf{a}(P)$.

Theorem 3.3.1. *For a given site $\mathbf{C} = (\mathcal{C}, \mathcal{J})$, the inclusion functor $i\colon \mathbf{Sh}(\mathbf{C}) \hookrightarrow \mathbf{PSh}(\mathcal{C})$ has a left adjoint*

$$\mathbf{a}\colon \mathbf{PSh}(\mathcal{C}) \to \mathbf{Sh}(\mathbf{C})$$

called the ***sheafification*** *or* ***associated sheaf functor****.*

In this section, we prove this theorem and give the construction of this functor.

Given a presheaf $P \in \mathbf{PSh}(\mathcal{C})$, we construct a new presheaf $P^\dagger$, which for an object K is calculated by

$$P^\dagger(K) = \varinjlim_{S\in\mathcal{J}(K)} \mathsf{Match}(S, P)$$

where $\mathsf{Match}(S, P)$ is the set of matching families of P for the cover S of K, their morphisms are those induced by reverse inclusion, and the colimit is taken over all covers of K.

We will see that the presheaf $P^\dagger$ is not necessarily a sheaf, but it is a **separated presheaf**, which means that every matching family has at most one amalgamation. In other words, $P^\dagger$ satisfies the uniqueness condition of sheaves, but not the existence condition. However, we will show that if P is separated, then $P^\dagger$ is in fact a sheaf, therefore we can construct the associated sheaf as $P^\# = (P^\dagger)^\dagger$.

In order to understand the functor $P^\dagger$, we can use the definition of colimits in the category **Set**. So, an element of $P^\dagger(K)$ is an equivalence class of matching families $\mathbf{x} = \{x_f \mid f\colon A \to K,\ f \in S\}$ (which means that $x_f \in P(A)$ and for a morphism $g\colon B \to A$, $P(g)(x_f) = x_{f\circ g}$). Two families $\mathbf{x} = \{x_f \mid f \in S\}$ and $\mathbf{y} = \{y_g \mid g \in R\}$ are equivalent, if there is a sieve $T \subset S \cap R$ with $T \in \mathcal{J}(K)$, such that $x_f = y_f$ for every $f \in T$.

$P^\dagger$ is a functor $\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$, since for a morphism $h\colon K \to K'$, we have

$$P^\dagger(h)(\{x_f \mid f \in S\} = \{x_{h\circ g} \mid g \in h^*(S)\}, \tag{3.1}$$

so $P^\dagger(h)$ is a morphism $P^\dagger(K') \to P^\dagger(K)$ and is well-defined because of the stability axiom.

Furthermore, the mapping $P \mapsto P^\dagger$ is a natural transformation, because for $\phi\colon P \to Q$, it is easy to deduce $\phi^\dagger\colon P^\dagger \to Q^\dagger$.

Finally, we can define the morphism $\eta\colon P \to P^\dagger$, which for $x \in P(K)$, gives $\eta_K(x) = \{P(f)(x) \mid f \in S_K^{\mathsf{max}}\}$.

Next, we introduce some lemmas.

Lemma 3.3.2. *A presheaf P is separated if and only if $\eta\colon P \to P^\dagger$ is monomorphism.*

Proof. η being a monomorphism is equivalent with all the corresponding maps η_K for $K \in \mathsf{Ob}(\mathcal{C})$ being monomorphisms. If P is separated, then for $x, y \in P(K)$, if $\eta_K(x) = \eta_K(y)$, then the matching family $\eta_K(x)$ has at most one amalgamation, which means $x = y$. Conversely, if η_K is monomorphism, then every matching family on the maximal sieve has at most one amalgamation, thus every matching family on any covering sieve (since they are contained in the maximal sieve), so P is separated. □

Lemma 3.3.3. *A presheaf P is a sheaf if and only if $\eta\colon P \to P^\dagger$ is isomorphism.*

Proof. The proof is like the previous one, but in addition to the uniqueness property, we also use the existence property. □

Lemma 3.3.4. *Any morphism from the presheaf P to a sheaf F, $\phi\colon P \to F$ factors uniquely through η (in other words, there is a morphism ϕ' with $\phi = \phi' \circ \eta$.*

$$\begin{array}{ccc} P & \xrightarrow{\eta} & P^\dagger \\ & \searrow^{\phi} & \downarrow{\scriptstyle \phi'} \\ & & F. \end{array}$$

Proof. As mentioned above, the elements of $P^\dagger(K)$ are matcing families of the form $\{x_f \mid f \in S\}$ for a cover S of the object K. If we have a morphism, $h\colon A \to K$, which belongs to S, then $\eta_A(x_h) = \{P(g)(x_h) \mid \mathsf{codom}(g) = A\}$. Moreover, $P^\dagger(h)(\{x_f \mid f \in S\} = \{x_{h\circ f'} \mid f' \in h^*(S)\}$. Since $h \in S$, then $h^*(S) = S_A^{\mathsf{max}}$. Combining these two facts we get that $\eta_A(x_h) = P^\dagger(h)(\{x_f \mid f \in S\})$.

Therefore, if the map ϕ' exists, we have that

$$F(h)(\phi'(\{x_f \mid f \in S\})) = \phi'(P^\dagger(h)(\{x_f \mid f \in S\})) = \phi'(\eta_A(x_h)) = \phi(x_h),$$

for all $h \in S$. Therefore, $\phi'(\{x_f \mid f \in S\})$ is the unique value y, with $F(h)(y) = \phi(x_h)$ for all $h \in S$. But this exists, since F is a sheaf and $\{\phi(x_h) \mid h \in S\}$ is a matching family. □

Lemma 3.3.5. *For any presheaf P, $P^\dagger$ is a separated presheaf.*

Proof. Let K be an object of $\mathcal{C}$ and S a cover of K. Also, assume we have two elements $\mathbf{x}, mathbfy \in P^\dagger(K)$ with $P^\dagger(h)(\mathbf{x}) = P^\dagger(h)(\mathbf{y})$ for every $h \in S$. In order to show that $P^\dagger$ is separated, it suffices to prove that $\mathbf{x} = \mathbf{y}$.

The elements $\mathbf{x}$ and $\mathbf{y}$ are equivalence classes of matching families, so we can express them as $\mathbf{x} = \{x_f \mid f \in R\}$ and $\mathbf{y} = \{y_g \mid g \in T\}$, where $R, T \in \mathcal{J}(K)$.

From $P^\dagger(h)(\mathbf{x}) = P^\dagger(h)(\mathbf{y})$, with $h\colon A \to K$, we deduce that there is a cover Q_h of A, such that $Q_h \subset h^*(R) \cap h^*(T)$ and $x_{h\circ t} = y_{h\circ t}$ for all $t \in Q_h$. By the transitivity axiom, we have that the cover $Q = \{h \circ t \mid h \in S, t \in Q\}$ is a cover of K and also $Q \subset R \cap T$. Therefore, the two matching families are equivalent, thus $\mathbf{x} = \mathbf{y}$. □

Lemma 3.3.6. *For any separated presheaf P, $P^\dagger$ is a sheaf.*

Proof. From the previous lemma, we have that for every matching family of $P^\dagger$ there is at most one amalgamation, so it suffices to prove the existence of an amalgamation. So, we assume an object K, a cover $S \in \mathcal{J}(K)$ and a matching family $\{\mathbf{x}_f \mid f \in S\}$. Here $\mathbf{x}_f \in P^\dagger(A)$ if $f\colon A \to K$, so it is itself an equivalence class of a matching family, $\mathbf{x}_f = \{x_{f,g} \mid g \in S_f\}$ for a cover $S_f \in \mathcal{J}(A)$.

Since $\{\mathbf{x}_f \mid f \in S\}$ is an equivalence class, for a morphism $h\colon A' \to A$, we have $P^\dagger(h)(\mathbf{x}_f) = \mathbf{x}_{f\circ h} \in P^\dagger(A')$. So, from relation 3.1, we get the equivalence of families

$$\{x_{f,h\circ g'} \mid g' \in h^*(S_f)\} \equiv \{x_{f\circ h,g} \mid g \in h^*(S_{f\circ h})\}.$$

Thus, from the definition, we have a cover of A', $T_{f,h} \subset h^*(S_f) \cap S_{f\circ h}$, such that $x_{f,h\circ g} = x_{f\circ h,g}$ for every $g \in T_{f,h}$.

Next, we set Q to be the sieve $\{f \circ g \mid f \in S, g \in S_f\}$, which is a cover of K from the transitivity axiom. We define a matching family for this cover, $\mathbf{y}$, as $y_{f\circ g} = x_{f,g}$ (as above), which is independent of f and g. To prove that, we use the last equation we proved. Assuming $f \circ g = f' \circ g'$ for $f, f' \in S$, $g \in S_f$ and $g' \in S_{f'}$, we take a morphism $k \in T_{f,g} \cap T_{f',g'}$ and we have

$$P(k)(x_{f,g}) = x_{f,g\circ k} = x_{f\circ g,k} = x_{f'\circ g',k} = x_{f',g'\circ k} = P(k)(x_{f',g'})$$

where we used that $\mathbf{x}_f, \mathbf{x}_{f'}$ are matching families. P is separated, hence we get that $x_{f,g} = x_{f',g'}$, which means that $\mathbf{y}$ is well defined.

Since for every $f \in S$, $\mathbf{x_f}$ is a matching family, we conclude that $\mathbf{y}$ is also a matching family for the cover $Q \in \mathcal{J}(K)$, so it is an element of $P^\dagger(K)$.

Now, we can show that $\mathbf{y}$ is the amalgamation of the matching family $\{\mathbf{x}_f \mid f \in S\}$. It suffices to show that for a morphism $f\colon A \to K$,

$$P^\dagger(\mathbf{y}) = \mathbf{x}_f \iff \{y_{f\circ h} \mid h \in f^*(Q)\} = \{x_{f,g} \mid g \in S_f\}$$

in $P^\dagger(A)$. But these two families are equivalent, since from the definition of Q, $S_f \subset f^*(Q)$ and for any $g \in S_f$, $y_{f\circ g} = x_{f,g}$ again from the definition. As a result, $P^\dagger$ is a sheaf. □

We now give the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. As we mentioned, $\mathbf{a}(P) = (P^\dagger)^\dagger$. From Lemmas 3.3.5 and 3.3.6, this gives us indeed a sheaf. Also, the map $P \xrightarrow{\eta_P} P^\dagger \xrightarrow{\eta_{P^\dagger}} P^\#$ is universal among maps of the presheaf P to sheaves from lemma 3.3.4, so $\mathsf{Nat}(P, F) \cong \mathsf{Nat}(\mathbf{a}(P), F)$, which means that $\mathbf{a}$ is left adjoint to the inclusion functor.

Chapter 4

Equivalence of Sheaves on Group Categories

In this section, we mainly present the result of [3], which is the equivalence of the category of G-sets (set representations) of a finite group G with the Grothendieck topos on the orbit category of the group G. We initially define these two notions and we continue by presenting the tools we use.

Definition 4.0.1 (G-Sets)**.** Let G be a group. A set X is called a **G-set**, if there exists a G-action $a\colon X \times G \to X$ that satisfies:

- the property of identity: $x \cdot e\colon = a(x, e) = x$ for all $x \in X$ and $e \in G$ the group identity, and
- the property of compatibility: $(x \cdot g) \cdot h\colon = a(a(x, g), h) = a(x, gh) = x \cdot (gh)$, for all $x \in X$ and all $g, h \in G$.

The category $\mathsf{Set} - G$ has objects all G-sets and morphisms all the functions between the sets that respect the group actions ($f(x \cdot g) = f(x) \cdot g$ for a function f and all $x \in X$, $g \in G$).

Definition 4.0.2 (Orbit category)**.** For a group G, the **orbit category**, denoted by $\mathcal{O}(G)$, is the category with:

- objects all the cosets of G, G/H for a subgroup H,
- morphisms all the group homomorphisms between cosets.

Since the orbits (G/H) can be considered as G-sets, it is easy to see that $\mathcal{O}(G)$ is a full subcategory of $\mathsf{Set} - G$.

4.1 Continuous and Cocontinuous Functors

Definition 4.1.1 (Restriction and Kan extensions)**.** Let $a\colon \mathcal{C} \to \mathcal{D}$ be a (covariant) functor. Then we can define the **restriction** along a, $\mathsf{Res}_a\colon \mathbf{PSh}(\mathcal{D}) \to \mathbf{PSh}(\mathcal{C})$ with $F \mapsto F \circ a$. This functor has two adjoint functors, the **left** and **right Kan extensions** along a, denoted $LK_a\colon \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$ and $RK_a\colon \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$, respectively.

In the theory of Kan extensions, many times we need the notion of comma category.

Definition 4.1.2 (Comma category)**.** Let $a\colon \mathcal{C} \to \mathcal{D}$ be a (covariant) functor and $d \in \mathsf{Ob}(\mathcal{D})$. Then the **comma category** d/a has objects $\{(t,x) \mid t\colon d \to a(x), x \in \mathsf{Ob}(\mathcal{C})\}$ and morphisms $(t,x) \xrightarrow{u} (t',x')$, which is derived by a morphism $u\colon x \to x'$ in $\mathcal{C}$, such that $t' = a(u) \circ t$.

We introduce two new type of functors between two sites, which will help us compare the sheaves on these two sites.

Definition 4.1.3 (Continuous and cocontinuous functors)**.** Given two sites $\mathbf{C} = (\mathcal{C}, \mathcal{J}_{\mathcal{C}})$ and $\mathbf{D} = (\mathcal{D}, \mathcal{J}_{\mathcal{D}})$, a functor $a\colon \mathcal{C} \to \mathcal{D}$ is called **continuous** if $\forall x \in \mathsf{Ob}(\mathcal{C})$ and $\forall S_x \in \mathcal{J}_{\mathcal{C}}(x)$, the image of the sieve $a(S_x)$ generates a covering sieve on $a(x)$ (**cover-preserving condition**) and $\forall d \in \mathsf{Ob}(\mathcal{D})$, the opposite comma category $(d/a)^{\mathsf{op}}$ is filtered (which means that every filtered diagram has a cocone–**flatness condition**).

A functor $b\colon \mathcal{C} \to \mathcal{D}$ is called **cocontinuous** if $\forall x \in \mathsf{Ob}(\mathcal{C})$ and $\forall S_{b(x)} \in \mathcal{J}_{\mathcal{D}}(b(x))$, $\exists S_x \in \mathcal{J}_{\mathcal{C}}(x)$ with $b(S_x) \subset S_{b(x)}$ (**cover-reflecting condition**).

Definition 4.1.4 (Geometric morphisms)**.** Let $\mathbf{C} = (\mathcal{C}, \mathcal{J}_{\mathcal{C}})$ and $\mathbf{D} = (\mathcal{D}, \mathcal{J}_{\mathcal{D}})$ be two sites. A **geometric morphism** between Grothendieck topoi $\Psi\colon \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{C})$ is given by a pair of functors (Ψ^*, Ψ_*) where $\Psi^*\colon \mathbf{Sh}(\mathbf{C}) \to \mathbf{Sh}(\mathbf{D})$ is left exact and left adjoint to $\Psi_*\colon \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{C})$. Ψ^* is called the **inverse image part** of the geometric morphism and Ψ_* is the **direct image part**.

Proposition 4.1.5. *A continuous functor $a\colon \mathcal{C} \to \mathcal{D}$ induces a morphism of topoi*

$$\Psi = (\Psi^*, \Psi_*)\colon \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{C}),$$

where $\Psi_* = \mathsf{Res}_a\colon \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{C})$ and $\Psi^* = \mathsf{LK}_a^{\#}\colon \mathbf{Sh}(\mathbf{C}) \to \mathbf{Sh}(\mathbf{D})$.

A cocontinuous functor $b\colon \mathcal{C} \to \mathcal{D}$ induces another morphism of topoi

$$\Phi = (\Phi^*, \Phi_*)\colon \mathbf{Sh}(\mathbf{C}) \to \mathbf{Sh}(\mathbf{D}),$$

where $\Phi_* = \mathsf{RK}_b\colon \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{C})$ and $\Phi^* = \mathsf{Res}_b^{\#}\colon \mathbf{Sh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{C})$.

See [4] in pages 563-565, 574.

4.2 The Transporter Category

From now on, we take a group G and we consider it as a category with one element ($\bullet$) and the elements of the group as the automorphisms of $\bullet$. We also take a G-poset $\mathcal{P}$, which we consider as a category with objects being the elements of the poset $\mathcal{P}$ and morphisms of the form $l_x^y\colon x \to y$ for $x, y \in \mathcal{P}$, if $x \leq y$. We denote x^g the image of the element $x \in \mathcal{P}$ under the action of the element $g \in G$ and also $(l_x^y)^g = l_{x^g}^{y^g}$ the image of the morphism l_x^y under the action of $g \in G$.

Definition 4.2.1 (Abstract transporter category)**.** Let $\mathcal{P}$ be a G-poset. Then the **(abstract) transporter category** $\mathcal{P} \rtimes G$ is a category with the same objects as $\mathcal{P}$ and with morphisms the formal products $l_x^y g\colon x^{g^{-1}} \to y$ where $l_x^y \in \mathsf{Mor}(\mathcal{P})$ and $g \in G$.

For convenience, we write $l^y_{x^g} g \colon x \to y$. For the composition, if we have $l^y_{x^g} g \colon x \to y$ and $l^x_{w^h} h \colon w \to x$, we get $(l^y_{x^g} g)(l^x_{w^h} h) = l^y_{w^{hg}}(hg) \colon w \to y$. Also, it is useful to see that any morphism $l^y_{x^g} g$ factorises as $(l^y_{x^g} 1)(l^{x^g}_{x^g} g)$.

We note that a transporter category does not always meet the Ore condition, for example when the poset has two initial (minimal) objects. For that reason, it is more convenient to consider posets with only one initial object.

Every transporter category has a corresponding natural functor $\pi \colon \mathcal{P} \rtimes G \to G$, such that $x \in \mathcal{P} \mapsto \bullet$ and $l^y_{x^g} g \mapsto g$.

Example 4.2.2. We can consider $\mathcal{P} = \mathcal{S}(G)$, where $\mathcal{S}(G)$ is the poset of all subgroups of G with action being the conjugation (for $H \in \mathcal{S}(G)$ and $g \in G$, $H^g = g^{-1}Hg$). Then $\mathcal{T}(G) = \mathcal{S}(G) \rtimes G$ is called the **(complete) transporter category**. In $\mathcal{T}(G)$, we have the morphism sets $\mathsf{Hom}_{\mathcal{T}(G)}(H, K) = \{l^K_{H^g} \colon H \to K \mid g^{-1}Hg \subset K\}$ for $H, K \in \mathcal{S}(G)$.

Proposition 4.2.3. *For a group G, the category $\mathcal{T}(G)$ has fibre products. Hence it satisfies the Ore condition and we can equip it with the atomic topology, resulting the site $\mathbf{T}G = (\mathcal{T}(G), \mathcal{J}_{at})$.*

Proof. We consider two morphisms $l^L_{H^g} g \colon H \to L$ and $l^L_{K^{g'}} g' \colon K \to L$. We will show that the subgroup $M = H^g \cap K^{g'}$ along with the morphisms $l^H_{M^{g^{-1}}} g^{-1} \colon M \to H$ and $l^K_{M^{g'^{-1}}} g'^{-1} \colon M \to K$ are the pullback ($l^H_{M^{g^{-1}}} \in \mathsf{Mor}(\mathcal{S}(G))$, since $M^{g^{-1}} = (H^g \cap K^{g'})^{g^{-1}} = H \cap K^{g'g^{-1}} \subset H$).

$$\begin{array}{ccc} M = H^g \cap K^{g'} & \xrightarrow{\;l^K_{M^{g'^{-1}}} g'^{-1}\;} & K \\ \Big\downarrow{\scriptstyle l^H_{M^{g^{-1}}} g^{-1}} & & \Big\downarrow{\scriptstyle l^L_{K^{g'}} g'} \\ H & \xrightarrow{\;l^L_{H^g} g\;} & L \end{array}$$

Consider another subgroup N with morphisms $l^H_{N^h} h \colon N \to H$ and $l^K_{N^{h'}} h' \colon N \to K$ that make the square below commute:

$$\begin{array}{ccc} N & \xrightarrow{\;l^K_{N^{h'}} h'\;} & K \\ \Big\downarrow{\scriptstyle l^H_{N^h} h} & & \Big\downarrow{\scriptstyle l^L_{K^{g'}} g'} \\ H & \xrightarrow{\;l^L_{H^g} g\;} & L \end{array}$$

This means that

$$(l^L_{H^g} g)(l^H_{N^h} h) = (l^L_{K^{g'}} g')(l^K_{N^{h'}} h') \iff l^L_{N^{hg}} hg = l^L_{N^{h'g'}} h'g',$$

so we have that $N^{hg} = N^{h'g'}$. From the morphism $l^H_{N^h} h$, we get that $N^h \subset H \Rightarrow N^{hg} \subset H^g$ and similarly $l^K_{N^{h'}} h'$ means that $N^{h'} \subset K \Rightarrow N^{h'g'} \subset K^{g'}$. Therefore, $N^{hg} = N^{h'g'} \subset H^g \cap K^{g'}$ and there is a morphism $l^M_{N^{hg}} hg \colon N \to M$. We can see that $(l^H_{M^{g^{-1}}} g^{-1})(l^M_{N^{hg}} hg) = l^H_{N^h} h$ and $(l^K_{M^{g'^{-1}}} g'^{-1})(l^M_{N^{hg}} hg) = (l^K_{M^{g'^{-1}}} g'^{-1})(l^M_{N^{h'g'}} h'g') = l^K_{N^{h'}} h'$, which means that $M = H^g \cap K^{g'}$ is universal, thus the pullback. □

It is also obvious that the single-object category G has fibre products $\bullet \times_\bullet \bullet = \bullet$, so we can also equip it with the atomic topology, which is the same as the trivial topology.

Definition 4.2.4 (Category extension)**.** We call **category extension**, a sequence of two functors

$$\mathcal{K} \xrightarrow{i} \mathcal{E} \xrightarrow{\rho} \mathcal{C},$$

which have the properties:

1. The three categories $\mathcal{K}, \mathcal{E}, \mathcal{C}$ have the same objects and i and ρ are identities on objects.
2. i is injective on morphisms and ρ is surjective on morphisms.
3. There are $u, v \in \mathsf{Mor}(\mathcal{E})$ with $\rho(u) = \rho(v)$, if and only if there is a unique morphism $w \in \mathsf{Mor}(\mathcal{K})$ such that $u \circ i(w) = v$.

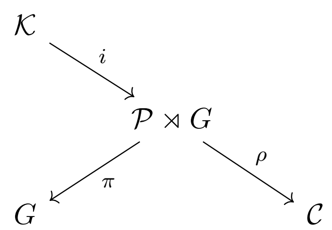
Proposition 4.2.5. *Let $\mathcal{K} \xrightarrow{i} \mathcal{E} \xrightarrow{\rho} \mathcal{C}$ be a category extension. Then, $\mathcal{K}$ has the form $\coprod_{x \in \mathsf{Ob}(\mathcal{C})} \mathcal{K}(x)$, where $\mathcal{K}(x)$ is category with only one object, x, and morphisms, all the automorphisms of x that are mapped to 1_x by $\rho \circ i$.*

Proof. We assume that there is at least a morphism in $\mathcal{K}$, $u \colon x \to y$ with $x \neq y$. Then, we have that $1_x \circ i(u) = i(u)$ and u is the only morphism in $\mathcal{K}$ with this property, since i is an injection in morphisms. As a result we have that $\rho(1_x) = \rho(i(u))$, which means that $x = y$. So, the only morphisms in $\mathcal{K}$ are automorphisms and we have that $\mathcal{K}$ is of the form $\coprod_{x \in \mathsf{Ob}(\mathcal{C})} \mathcal{K}(x)$.

Also, from the relation $\rho(1_x) = \rho(i(u))$, we get that $\rho \circ i(u) = 1_x$, so every automorphism in $\mathcal{K}$ is mapped to 1_x by $\rho \circ i$. 1_x and $i(u)$ satisfy the condition of property 3, thus there is a unique $u' \in \mathsf{Mor}(\mathcal{K})$, such that $1_x = i(u) \circ i(u') = i(u \circ u') \Rightarrow u \circ u' = 1_x$. So, u has the right inverse u' and similarly u' has a right inverse which must be equal to u. Therefore, every morphism has an inverse, and $\mathcal{K}(x)$ is a group. □

We can see from the first isomorphism theorem that if $\mathcal{K}(x)$ is always trivial, then $C \cong P \rtimes G$.

We consider a category extension $\mathcal{K} \xrightarrow{i} \mathcal{P} \rtimes G \xrightarrow{\rho} \mathcal{C}$ with the natural functor to G:



We can consider $\mathcal{K}(x)$ to be subgroups of $\mathsf{Aut}_{\mathcal{P} \rtimes G}(x)$ and $\mathcal{K} = \coprod_{x \in \mathsf{Ob}(\mathcal{C})} \mathcal{K}(x)$.

In this picture, we know that $\mathbf{PSh}(G) = \mathsf{Set} - G$. Also, for a G-Set M, we get the presheaf $\kappa_M = \mathsf{Res}_\pi(M)$, which is constant on objects (sends every object of $\mathcal{P} \rtimes G$ to the set M). Furthermore, the right and left Kan extensions along π are isomorphic to limits and colimits respectively (see [5]).

Our purpose is to construct the respective picture for sheaves.

At first, we need some sort of Grothendieck topology for the categories $\mathcal{P} \rtimes G$ and $\mathcal{C}$. We introduce the next lemma to see how we can equip these categories with the atomic topology.

Lemma 4.2.6. *Suppose a group G and a G-poset $\mathcal{P}$ with an initial object. Then $\mathcal{P} \rtimes G$ satisfies the Ore condition, thus it can be equipped with the atomic topology. Moreover, $\forall x \in \mathsf{Ob}(\mathcal{P} \rtimes G)$, there is a unique minimal non-empty sieve.*

Proof. We call the unique initial object of $\mathcal{P}$, x_0. If for any $g \in G$, $x_0^g \neq x_0$, then $x_0 < x_0^g \Rightarrow x_0^{g^{-1}} < x_0$, which is a contradiction, therefore $x_0^g = x_0$ for all $g \in G$.

We shall prove that $\mathcal{P} \rtimes G$ satisfies the Ore condition. Assume two morphisms $l_{y^g}^x g \colon y \to x$ and $l_{z^h}^x h \colon z \to x$. Then, we can complete the Ore square with the morphisms $l_{x_0}^y g^{-1} \colon x_0 \to y$ and $l_{x_0}^z h^{-1} \colon x_0 \to z$, since $(l_{y^g}^x g)(l_{x_0}^y g^{-1}) = l_{x_0}^x 1 = (l_{z^h}^x h)(l_{x_0}^z h^{-1})$. Therefore, $\mathcal{P} \rtimes G$ can be given the atomic topology.

Next, let S be a non-empty sieve on $x \in \mathsf{Ob}(\mathcal{P})$, which contains the morphism $l_{y^g}^x g \colon y \to x$. We can take the morphism $l_{x_0}^y g^{-1} \colon x_0 \to y$ (it is a morphism since $x_0 \leq y$) and composing we get $(l_{y^g}^x g)(l_{x_0}^y g^{-1}) = l_{x_0}^x 1$, so $l_{x_0}^x 1 \in S$. Furthermore, for any $g' \in G$, we have that $(l_{x_0}^x 1)(l_{x_0}^{x_0} g') = l_{x_0}^x g' \in S$. These morphisms are independent from the sieve S, so the set $S_x^{\mathsf{min}} = \{l_{x_0}^x g \mid g \in G\}$ is contained in any sieve on x and it is itself a sieve, therefore it is minimal among the non-empty sieves on x (and unique).

If we want to characterise it as a subfunctor of $S_x^{\mathsf{max}} = \mathsf{Hom}_{\mathcal{P} \rtimes G}(-, x)$, we can write it as:

$$S_x^{\mathsf{min}}(y) = \begin{cases} \mathsf{Hom}_{\mathcal{P} \rtimes G}(x_0, x), & \mathsf{if} y = x_0 \\ \emptyset & \mathsf{otherwise.} \end{cases}$$

□

It is interesting to see that $\mathsf{Hom}_{\mathcal{P} \rtimes G}(x_0, x)$ is isomorphic with G as a G-set.

From now on, we denote $\mathbf{P}G$ the site $(\mathcal{P} \rtimes G, \mathcal{J}_{at})$, if $\mathcal{P}$ has an initial object. $\mathbf{T}G = (\mathcal{T}(G), \mathcal{J}_{at})$ is a special case of $\mathbf{P}G$.

Corollary 4.2.7. *Suppose a group G and a G-poset $\mathcal{P}$ with an initial object. We also have a covariant functor $\rho \colon \mathcal{P} \rtimes G \to \mathcal{C}$, which is identity on objects and surjective on morphisms. Then $\mathcal{C}$ satisfies the Ore condition, thus it can be equipped with the atomic topology. Moreover, $\forall x \in \mathsf{Ob}(\mathcal{C})$, there is a unique minimal non-empty sieve.*

Proof. Since ρ is identity on objects, we identify the objects of $\mathcal{P} \rtimes G$ and $\mathcal{C}$. Again, we take two morphisms $y \to x$ and $z \to x$ in $\mathcal{C}$ and since ρ is surjective on morphisms, we can take one of their preimages in $\mathcal{P} \rtimes G$. The morphisms taken can complete a commutative square from the previous Lemma, so taking the image of the square along ρ, we get a commutative square for the two initial morphisms in $\mathcal{C}$. Hence, $\mathcal{C}$ satisfies the Ore condition.

We consider a non-empty sieve S on the object x of $\mathcal{C}$. We have that $\rho^{-1}(S) = \{u \in \mathsf{Mor}(\mathcal{P} \rtimes G) \mid \rho(a) \in S\}$, which is a non-empty sieve (ρ is surjective on morphisms). Also, if S' is a non-empty sieve on x in the category $\mathcal{P} \rtimes G$, then $\rho(S')$ is a sieve on x in the category $\mathcal{C}$. Taking the unique minimal sieve S_x^{min} from the previous lemma, we get that $S_x^{\mathsf{min}} \subset \rho^{-1}(S) \Rightarrow \rho(S_x^{\mathsf{min}}) \subset S$ for any non-empty sieve S. Therefore, the sieve $\rho(S_x^{\mathsf{min}})$ is minimal among non-empty sieves on $x \in \mathsf{Ob}(\mathcal{C})$, and unique.

Again, if we want to characterise it as a subfunctor of $S_x^{\mathsf{max}} = \mathsf{Hom}_{\mathcal{C}}(-, x)$, we can write it as:

$$S_x^{\mathsf{min}}(y) = \begin{cases} \mathsf{Hom}_{\mathcal{C}}(x_0, x), & \mathsf{if} y = x_0 \\ \emptyset & \mathsf{otherwise.} \end{cases}$$

□

Proposition 4.2.8. *Suppose a group G and a G-poset $\mathcal{P}$ with an initial object x_0. We also have the category extension $\mathcal{K} \xrightarrow{i} \mathcal{P} \rtimes G \xrightarrow{\rho} \mathcal{C}$. If we give the atomic topology on $\mathcal{P} \rtimes G$ and $\mathcal{C}$, then the sheafification of $F \in \mathbf{PSh}(\mathcal{C})$ is the fixed-point sheaf $\mathcal{F}_{F(x_0)} \in \mathbf{Sh}(\mathbf{C})$, which is the sheaf with $\mathcal{F}_{F(x_0)}(x) = F(x_0)^{\mathcal{K}(x)}$. As a result, we have that all the sheaves on $\mathbf{C}$ are of the form $\mathcal{F}_M$ and all the sheaves on $\mathbf{P}G$ are the constant sheaves κ_M, with M being any G-set.*

Proof. We consider a presheaf F on $\mathcal{C}$ and we apply the sheafification procedure to it, constructing $F^\dagger \in \mathbf{PSh}(\mathcal{C})$, where

$$F^\dagger(x) = \varinjlim_{S \in \mathcal{J}(x)} \mathsf{Nat}(S, F), \quad \forall x \in \mathsf{Ob}(\mathcal{C}).$$

According to the process, if $F^\dagger$ is not a sheaf, then we repeat, getting $F^\# = (F^\dagger)^\dagger \in \mathbf{Sh}(\mathbf{C})$.

However, by Corollary 4.2.7, for every $x \in \mathsf{Ob}(\mathcal{C})$, there is always a unique minimal non-empty sieve $S_x^{\mathsf{min}} \in \mathcal{J}(x)$. Since there is a morphism from S_x^{min} to any other sieve S, from naturality we have that all the morphisms $\mathsf{Nat}(S, F) \to \mathsf{Nat}(S_x^{\mathsf{min}}, F)$ form a cone and $\mathsf{Nat}(S_x^{\mathsf{min}}, F)$ is an element of the cone, thus concluding that $F^\dagger(x) \cong \mathsf{Nat}(S_x^{\mathsf{min}}, F)$.

Moreover, since the image under $\rho \circ i$ of any morphism in $\mathcal{K}(x)$ is 1_x, we get that $\mathsf{Mor}(\mathcal{K}(x)) = \mathsf{ker}\rho$, where ρ is considered as a function between the morphisms groups. ρ is also surjective, hence we get

$$\mathsf{Hom}_{\mathcal{C}}(x_0, x) \cong \mathsf{Hom}_{\mathcal{P} \rtimes G}(x_0, x)/\mathcal{K}(x),$$

which can be considered as the G-set $G/\mathcal{K}(x)$.

Since S_x^{min} is defined only on x_0 and as a sieve it returns $\mathsf{Hom}_{\mathcal{C}}(x_0, x)$, $\mathsf{Nat}(S_x^{\mathsf{min}}, F) \cong \mathsf{Hom}_G(G/\mathcal{K}(x), F(x_0)) \cong F(x_0)^{\mathcal{K}(x)}$. Concluding, we get $F^\dagger(x) \cong F(x_0)^{\mathcal{K}(x)}$, $\forall x \in \mathsf{Ob}(\mathcal{C})$, so $F^\dagger \cong \mathcal{F}_{F(x_0)}$. $\mathcal{F}_{F(x_0)}$ is already a sheaf, so it is indeed the sheafification of F.

Furthermore, $F(x_0)$ is always a G-set. Conversely, if M is a G-set, we can define a presheaf F_M with $F(x_0) = M$ and $F(x) = \emptyset$ otherwise, which has sheafification $\mathcal{F}_M$.

In the case of $\mathcal{P} \rtimes G$, we identify it with the category $\mathcal{C}$. This means that ρ is identity in morphisms, so $\mathcal{K}(x) = \mathsf{ker}\rho = 1$ for all $x \in \mathsf{Ob}(\mathcal{C})$. Therefore, for any presheaf F on $\mathcal{P} \rtimes G$, we have the sheafification $F^\dagger(x) \cong F(x_0)^{\mathcal{K}(x)} = F(x_0)$, thus the constant sheaf $\kappa_{F(x_0)}$. □

Proposition 4.2.9. *Suppose a group G and a G-poset $\mathcal{P}$ with an initial object. Let $\pi \colon \mathcal{P} \rtimes G \to G$ be the natural functor to G and $\rho \colon \mathcal{P} \rtimes G \to \mathcal{C}$ be the second functor on a category extension (identity on objects and surjective on morphisms). Then, giving $\mathcal{P} \rtimes G$, G and $\mathcal{C}$ the atomic topologies, we have:*

1. *π is continuous and cocontinuous*

2. *ρ is cocontinuous.*

Proof. π is continuous: The image of $S_x^{\mathsf{min}} \in \mathcal{J}(x)$ under π is the group G, which means all the automorphisms on $\bullet$. This is the only covering sieve on $\bullet$. Since S_x^{min} is contained in any sieve on x, $\pi(S_x)$ is a covering sieve on $\bullet$, for any $S_x \in \mathcal{J}(x)$. Also, the category $\bullet/\pi$ has objects (t, x) with t being a morphism in G, $\bullet \to \pi(x) = \bullet$, so t is identified with some $g \in G$. If we have a morphism in $\bullet/\pi$, $u\colon (g, x) \to (g', x')$, then there is a morphism $u\colon x \to x'$ in $\mathcal{P} \rtimes G$, such that $g' = \pi(u) \circ g \Rightarrow \pi(u) = g'g^{-1}$. As a result, $u = l^{x'}_{x^{g'g^{-1}}} g'g^{-1}\colon x \to x'$. It is easy to see that all the objects of the form (g, x_0), for any $g \in G$ and for x_0 being the initial element of $\mathcal{P}$, are isomorphic with each other and initial. Therefore, $(\bullet/\pi)^{\mathsf{op}}$ has a terminal object, thus π is continuous.

π is cocontinuous: For any $x \in \mathsf{Ob}(\mathcal{P} \rtimes G)$, $\pi(x) = \bullet$ and there is only one sieve on $\bullet$, S, the sieve of all automorphisms. If we take the minimal sieve S_x^{min} on x, then its image under π is exactly the sieve S.

ρ is cocontinuous: As we have shown in Corollary 4.2.7, for any sieve S on $x \in \mathsf{Ob}(\mathcal{C})$, $\rho(S_x^{\mathsf{min}}) \subset S$ and from the atomic topology $S_x^{\mathsf{min}} \in \mathcal{J}(x)$ in $\mathcal{P} \rtimes G$.

□

The previous proposition shows how we use the atomic topology in order to have continuous and cocontinuous functors. Now using their properties, we get the following result:

Corollary 4.2.10. *Suppose a group G and a G-poset $\mathcal{P}$ with an initial object. The natural functor $\pi\colon \mathcal{P} \rtimes G \to G$ induces a morphism of topoi:*

$$\Xi = (\Xi^*, \Xi_*)\colon \mathbf{Sh}(\mathbf{G}) \to \mathbf{Sh}(\mathbf{P}G),$$

and another morphism of topoi:

$$\Pi = (\Pi^*, \Pi_*)\colon \mathbf{Sh}(\mathbf{P}G) \to \mathbf{Sh}(\mathbf{G}).$$

If we have a category extension $\mathcal{K} \xrightarrow{i} \mathcal{P} \rtimes G \xrightarrow{\rho} \mathcal{C}$, then the functor ρ induces a morphism of topoi:

$$\Theta = (\Theta^*, \Theta_*)\colon \mathbf{Sh}(\mathbf{P}G) \to \mathbf{Sh}(\mathbf{C}).$$

Proof. We define $\Xi_* = \mathsf{Res}_\pi\colon \mathbf{Sh}(\mathbf{G}) \to \mathbf{Sh}(\mathbf{P}G)$ and $\Xi^*\colon \mathbf{Sh}(\mathbf{P}G) \hookrightarrow \mathbf{PSh}(\mathcal{P} \rtimes G) \xrightarrow{LK_\pi} \mathbf{PSh}(G) = \mathbf{Sh}(\mathbf{G})$, which form an adjoint pair. Also, the forgetful functor is left exact and $LK_\pi = LK_\pi^{\#}$ is exact, since $LK_\pi \cong \varinjlim_{\mathcal{P}}$ and $\mathcal{P}$ has an initial object x_0, so Ξ^* is left exact. Therefore, Ξ is a morphism of topoi.

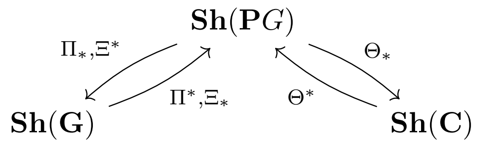
We note that the sheaf $\Xi_*(M)$ is the sheaf that sends an object x of $\mathbf{Sh}(\mathbf{P}G)$ to $\bullet$ through π and afterwards to M (the sheaf on $\mathbf{G}$). This is exactly the sheaf κ_M, so $\Xi_*(M) = \kappa_M$. Furthermore, $\Xi^*(\kappa) = \kappa(x_0)$.

Due to the cocontinuity of π, we get that $RK_\pi F \cong \varprojlim_{\mathcal{P}} F$. Therefore, we put $\Pi_* = RK_\pi$ and $\Pi^* = \mathsf{Res}_\pi$.

Similarly, from the cocontinuity of ρ we have that that $RK_\rho F \in \mathbf{Sh}(\mathbf{C})$ for any $F \in \mathbf{Sh}(\mathbf{P}G)$. Then we can put $\Theta_* = RK_\rho$ and $\Theta^* = \mathsf{Res}_\rho^{\#}$. The functor Θ^* is exact. □

4.3 Equivalence of Topoi

Finally, we have the picture:



Theorem 4.3.1. *Suppose a group G, a G-poset $\mathcal{P}$ with an initial object and a category extension $\mathcal{K} \to \mathcal{P} \rtimes G \to \mathcal{C}$. We equip the categories G, $\mathcal{P} \rtimes G$ and $\mathcal{C}$ with the atomic topology. Then, we have:*

1. *The topoi $\mathbf{Sh}(\mathbf{G})$ and $\mathbf{Sh}(\mathbf{C})$ are both equivalent to $\mathbf{Sh}(\mathbf{P}G)$, given by the topoi morphisms Ξ and Θ respectively.*

2. *The topoi morphism $Y = \Theta \circ \Xi \colon \mathbf{Sh}(\mathbf{G}) \to \mathbf{Sh}(\mathbf{C})$ is an equivalence.*

Proof. The sheaves of the topos $\mathbf{Sh}(\mathbf{G})$ are all the G-sets, while the sheaves on the site $\mathbf{P}G$ are the constant sheaves κ_M and the sheaves on $\mathbf{C}$ are the fixed-point sheaves $\mathcal{F}_M$, where M is running over all G-sets (Proposition 4.2.8). So, the morphisms Ξ and Θ are essentially surjective.

Also, using calculations, for a G-set M, we have

$$\Theta_* \circ \Theta^*(\mathcal{F}_M) = \Theta_*(\kappa_M) = \mathcal{F}_M,$$

$$\Theta^* \circ \Theta_*(\kappa_M) = \Theta^*(\mathcal{F}_M) = \kappa_M,$$

$$\Xi^* \circ \Xi_*(M) = \Xi^*(\kappa_M) = \kappa_M(x_0) = M,$$

$$\Xi_* \circ \Xi^*(\kappa_M) = \Xi_*(M) = \kappa_M.$$

So, the units and counits of the adjunctions Θ and Ξ are isomorphisms, thus they are equivalences. In the same way, we can show that Π is also an equivalence.

We set $Y = (Y^*, Y_*) \colon \mathbf{Sh}(\mathbf{G}) \to \mathbf{Sh}(\mathbf{C})$, where $Y_* = \mathsf{RK}_\rho \circ \mathsf{Res}_\pi$ and $Y^* = \mathsf{LK}_\pi \circ \mathsf{Res}_\rho^{\#}$. Since Θ and Ξ are equivalences, so is Y.

□

We can use this theorem for the complete transformer category, so we set $\mathcal{P} = \mathcal{S}(G)$ and we can choose $\mathcal{C} = \mathcal{O}(G)$.

Corollary 4.3.2. *For a finite group G, if we equip the orbit category $\mathcal{O}(G)$ with the atomic topology, making the site $\mathbf{O}G = (\mathcal{O}(G), \mathcal{J}_{\mathsf{at}})$, we have the equivalence:*

$$\mathbf{Sh}(\mathbf{O}G) \cong \mathsf{Set} - G.$$

In a similar way, we can apply this result to different posets, so getting different orbit categories. Below are given some examples, which are presented in the referenced paper:

- $\mathcal{S}_p(G)$ (all the subgroups for a prime p), $\mathcal{T}_p(G)$ (p-transporter category) and $\mathcal{O}_p(G)$ (p-orbit category)

- $\mathcal{S}_b(G)$ (all b-Brauer pairs for a p-block b of kG), $\mathcal{T}_b(G)$ (b-transporter category) and $\mathcal{O}_b(G)$ (b-orbit category)
- $\mathcal{P}(G)$ (all local pointed groups over kGb), $\mathcal{T}_{LP}(G)$ (p-local transporter category) and $\mathcal{O}_{LP}(G)$ (p-local orbit category)

Using Theorem 4.3.1, we can have not only the equivalence of the category of G-Sets with $\mathbf{Sh}(\mathbf{O}G)$, but also with $\mathbf{Sh}(\mathbf{O}_pG), \mathbf{Sh}(\mathbf{T}_pG), \mathbf{Sh}(\mathbf{O}_bG)$, etc.

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