Fundamental Group and Universal Covering of Lie Groups

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February 2023

The purpose of this project is to introduce the notions of the fundamental group and the universal covering, especially in the case of Lie groups, which combine geometric with algebraic properties. In particular, after introducing Lie groups and Lie algebras, the fundamental group and covering maps, we prove that all connected Lie groups that have the same Lie algebra are the quotient of the unique simply-connected Lie group that corresponds to this algebra, with any of its discrete normal subgroups. This result is remarkable, since it links the geometric structure of a Lie group with its algebraic structure. Finally, we apply these results to prove that $SU(2)$ is the universal covering of $SO(3)$, which is generated by the quotient of $SU(2)$ with its center.

1 Lie Groups and Lie Algebras

1.1 Lie Groups

Definition 1.1 (**Lie Groups**)**.** A **Lie group** is a smooth manifold G together with a group structure on it such that the map

$$G \times G \to G \quad (g,h) \mapsto g \cdot h^{-1}$$

is smooth.

Examples:

1. $(\mathbb{R}, +)$ is a commutative Lie group.
2. The unit circle on the complex plane is also a commutative Lie group.
3. The **general linear group** $\mathbf{GL}(n, \mathbb{K})$ is the group of invertible $n \times n$ matrices with entries in the field $\mathbb{K} = \mathbb{C}, \mathbb{R}$. $\mathbf{GL}(n, \mathbb{K})$ is a Lie group with the group operation of known matrix multiplication.

Next, we define two important subgroups of $\mathbf{GL}(n, \mathbb{K})$ that will concern us in this project.

4. The **orthogonal group** $O(n)$ is the subgroup of $\mathbf{GL}(n, \mathbb{R})$ with matrices that satisfy
$$M \cdot M^T = I.$$
$O(n)$ is the group of distance-preserving transformations of a Euclidean space of dimension n that preserve a fixed point. The dimension of $O(n)$ is $\frac{n(n-1)}{2}$.

   The **special orthogonal group** $SO(n)$ is a normal subgroup of $O(n)$, such that
$$SO(n) = \{M \in O(n) : \det M = 1\}.$$
$SO(n)$ is also a Lie group with $\dim SO(n) = \dim O(n) = \frac{n(n-1)}{2}$. In dimensions 2 and 3, $SO(2)$ and $SO(3)$ are the sets of rotations around a point and a line, respectively.

5. The **unitary group** $U(n)$ is the subgroup of $\mathbf{GL}(n, \mathbb{C})$ with matrices that satisfy
$$M \cdot M^* = I,$$
where M^* denotes the conjugate transpose of M. The dimension of $U(n)$ is n^2.

   The **special unitary group** $SU(n)$ is a normal subgroup of $U(n)$, such that
$$SU(n) = \{M \in U(n) : \det M = 1\}.$$
An alternative definition for this is that $SU(n)$ is the kernel of the group homomorphism $\det : U(n) \to U(1) \cong \mathbb{S}^1$. The dimension of $SU(n)$ is thus $n^2 - 1$.

Definition 1.2 (**Lie group homomorphism**)**.** If G and H are Lie groups, a **Lie group homomorphism** from G to H is a smooth map $F : G \to H$ that is also a group homomorphism.

A Lie group homomorphism $F : G \to H$ is called a **Lie group isomorphism**, if it is also a diffeomorphism, which means that it has an inverse which is also a Lie group homomorphism. In this case we say that G and H are isomorphic Lie groups.

Theorem 1.3. *Every Lie group homomorphism $F : G \to H$ has constant rank.*

Proof. Let $e \in G$ denote the identity element of G and $g_0 \in G$ be an arbitrary element. Then, we can define a smooth action called **left translation** L_g as $L_g(h) = g \cdot h$, which is a diffeomorphism. Using this, we will show that $(F_*)_{g_0}$ has the same rank as $(F_*)_e$.

We have that $\forall g \in G$:
$$F \circ L_{g_0}(g) = F(g_0 g) = F(g_0)F(g) = L_{F(g_0)}F(g) = L_{F(g_0)} \circ F(g)$$
Shortly, $F \circ L_{g_0} = L_{F(g_0)} \circ F$. Taking the differential of both sides at e, we get:
$$(F_*)_{g_0} \circ (L_{g_0})_e = (L_{F(g_0)*})_{F(e)} \circ (F_*)_e$$

But L_g is a diffeomorphism, so its differential is isomorphism. Furthermore, we know that compositition with an isomorphism does not change the rank, so finally we get: $rank(F_*)_{g_0} = rank(F_*)_e$, as we wanted. □

Lemma 1.4. *Every continuous homomorphism of Lie groups is smooth.*

Proof. Let $\Phi : G \to H$ be a continuous homomorphism. Then $\Gamma_\Phi = \{(g, \Phi(g) : g \in G\}$ is a Lie subgroup of $G \times H$.

The projection $p : \Gamma_\Phi \xrightarrow{i} G \times H \xrightarrow{pr_1} G$ is a bijective, smooth Lie homomorphism, so from Theorem 1.3, p_* has constant rank. This means that p is a diffeomorphism, thus $\Phi = pr_2 \circ p^{-1}$ is smooth. □

1.2 Lie Algebras

In the previous proof we defined the left translation diffeomorphism L_g of a Lie group G. A vector field $X \in \mathfrak{X}(G)$ is then called **left-invariant**, if it is invariant under all left translations, which means

$$d(L_g)_h(X_h) = X_{gh} \quad \forall g, h \in G \Rightarrow (L_g)_* X = X \quad \forall g \in G$$

Since $(L_g)_*(aX + bY) = a(L_g)_*X + b(L_g)_*Y$, the set of all smooth left-invariant vector fields on G is a linear subspace of $\mathfrak{X}(G)$. It is also closed under Lie brackets, because

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y],$$

where the first equality comes from the naturality of the Lie bracket. So, we have the following definition:

Definition 1.5 (**Lie Algebra**)**.** The set of all left-invariant vector fields of G is denoted by $Lie(G)$ or $\mathfrak{g}$ and it is called the **Lie algebra** of the Lie Group G. It is a subalgebra of $\mathfrak{X}(G)$, so the Lie bracket satisfies the properties of bilinearity, antisymmetry and the Jacobi identity.

Proposition 1.6. *The dimension of $Lie(G)$ is equal with the dimension of the corresponding group G.*

Proof. We will show equivalently that $Lie(G) \cong T_eG$. We define the evaluation map $\epsilon : Lie(G) \to T_eG$ by $\epsilon(X) = X_e$. This map is linear over $\mathbb{R}$ by definition and it is injective, because if $\epsilon(X) = X_e = 0$, then $X_g = X_{ge} = d(L_g)_e(X_e) = 0$, $\forall g \in G$, so $X = 0$.

It suffices to show that ϵ is surjective. Let $v \in T_eG$. We define a new vector field v^L on G by

$$v^L_g = d(L_g)_e(v).$$

The vector field v^L is smooth, since L_g is smooth. It is sufficient to show that $v^L f$ is smooth for any smooth function $f \in C^\infty(G)$. We choose a smooth curve $\gamma : (-\delta, \delta) \to G$, such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then $\forall g \in G$:

$$(v^L f)(g) = v^L_g f = d(L_g)_e(v) f = v(f \circ L_g) = \gamma'(0)(f \circ L_g) = \frac{d}{dt}(f \circ L_g \circ \gamma)(t)|_{t=0}.$$

Defining $\phi : (-\delta, \delta) \times G \to \mathbb{R}$ by $\phi(t, g) = f \circ L_g \circ \gamma(t) = f(g\gamma(t))$, we have that $v^L f(g) = \partial\phi/\partial t(0, g)$. Since f, γ and the group multiplication are smooth, we conclude that $\partial f/\partial t(0, g)$ is smooth on g, so $v^L f$ is smooth.

Furthermore, since we have $L_h \circ L_g = L_{hg}$, we get that

$$d(L_h)_g(v_g^L) = d(L_h)_g \circ d(L_g)_e(v) = d(L_h \circ L_g)_e(v) = d(L_{hg})_e(v) = v_{hg}^L.$$

As a result $(L_h)_* v^L = v^L$, $\forall h \in G$, so v^L is left-invariant.

Finally, we have that $v^L \in Lie(G)$ and $\epsilon(v^L) = v_e^L = v$ for any $v \in T_eG$, so ϵ is surjective. □

The isomorphism $\mathbf{Lie(G)} \cong \mathbf{T_eG}$ is very important, so from now on, we will use it in many proofs.

2 Homotopy and Fundamental Group

2.1 Definition

Definition 2.1 (**Homotopy**). We define as a **path** in a topological space X a continuous map $\gamma : [0, 1] \to X$. A **homotopy** of paths in X is a family of paths $\gamma_t : [0, 1] \to X$, $0 \leq t \leq 1$, such that:

1. The endpoints $\gamma_t(0) = x_0$ and $\gamma_t(1) = x_1$ are independent of t.
2. The associated map $F : [0, 1] \times [0, 1] \to X$ defined by $\Gamma(s, t) = \gamma_t(s)$ is continuous.

If two paths γ_0 and γ_1 are connected in this way by a homotopy γ_t, they are said to be **homotopic**. We denote that as $\gamma_0 \sim \gamma_1$.

Proposition 2.2. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation. The equivalence class of a path γ under the equivalence relation of homotopy is denoted $[\gamma]$ and called the* ***homotopy class*** *of γ.*

Proof. The property of reflexivity is evident, because $\gamma \sim \gamma$ by the constant homotopy $\gamma_t = \gamma$. Symmetry is also easy, since if $\gamma_0 \sim \gamma_1$ by the homotopy γ_t, then $\gamma_1 \sim \gamma_0$ by the homotopy γ_{1-t}.

For transitivity, we assume that $\gamma_0 \sim \gamma_1$ by γ_t and $\eta_0 \sim \eta_1$ by η_t with $\gamma_1 = \eta_0$. Then $\gamma_0 \sim \eta_1$ by the homotopy h_t that is equal with γ_{2t} for $t \in [0, 1/2]$ and with η_{2t-1} for $t \in [1/2, 1]$. The homotopy h_t is continuous, since both of its components are continuous and they agree on the intersection $t = 1/2$ from $\gamma_1 = \eta_0$. □

We can generalise the notion of homotopy from paths to general maps.

Definition 2.3 (**Homotopic Maps**). Let X, Y be two topological spaces. Two continuous maps $f_0, f_1 : X \to Y$ are said to be homotopic if there exists a continuous map

$$F : [0, 1] \times X \to Y \quad (t, x) \mapsto F_t(x),$$

such that, $F_i \equiv f_i$ for $i = 0, 1$. Similarly to paths, we denote that as $f_0 \sim f_1$.

2.2 Fundamental Group

We begin with the definition of an operation on paths, so as to see how we can create a group in a space, which characterises it.

Definition 2.4. Given two paths $\gamma, \eta : [0,1] \to X$ such that $\gamma(1) = \eta(0)$, we define the **composition** or **product path** $\gamma \cdot \eta$ that traverses first γ and then η, defined by the formula

$$(\gamma \cdot \eta)(s) = \begin{cases} \gamma(2s), & 0 \leq s \leq 1/2 \\ \eta(2s-1), & 1/2 \leq s \leq 1 \end{cases}$$

It is easy to see that if $\gamma_0 \sim \gamma_1$ and $\eta_0 \sim \eta_1$, then $\gamma_0 \cdot \eta_0 \sim \gamma_1 \cdot \eta_1$ using the homotopy $\gamma_t \cdot \eta_t$.

Definition 2.5 (**Fundamental Group**)**.** A path $\gamma : [0,1] \to X$ with the same starting and ending point $\gamma(0) = \gamma(1) = x_0 \in X$ is called a **loop**, and the common starting and ending point x_0 is called the **basepoint** of the loop.

The set of all homotopy classes of loops with the basepoint x_0 is denoted $\pi_1(X, x_0)$ and it is a group with respect to the product $[\gamma] \cdot [\eta] = [\gamma \cdot \eta]$. The group $\pi_1(X, x_0)$ is called the **fundamental group** of X at the basepoint x_0.

Proof. The product $[\gamma] \cdot [\eta] = [\gamma \cdot \eta]$ is well-defined, because if $\gamma_0 \sim \gamma$ and $\eta_0 \sim \eta$, then $\gamma_0 \cdot \eta_0 \sim \gamma \cdot \eta$ using the homotopy $\gamma_t \cdot \eta_t$. For the associativity property, if we are given γ, η, μ paths with $\gamma(1) = \eta(0)$ and $\eta(1) = \mu(0)$, then it is easy to construct a homotopy between $(\gamma \cdot \eta) \cdot \mu$ and $\gamma \cdot (\eta \cdot \mu)$, so $(\gamma \cdot \eta) \cdot \mu \sim \gamma \cdot (\eta \cdot \mu)$. Restricting that to loops at basepoint x_0, we get that $\pi_1(X, x_0)$ is associative.

We now define the identity of the group, which is the constant loop $c : [0,1] \to X$, $c(t) = x_0$. Then, for any path $\gamma : [0,1] \to X$, $\gamma \cdot c \sim c \cdot \gamma$ via a reparametrisation of the paths.

Finally, for a path $\gamma : [0,1] \to X$ from x_0 to x_1, we define the inverse path γ^- from x_1 to x_0 by $\gamma^-(s) = \gamma(1-s)$. We see that $\gamma \cdot \gamma^-$ is homotopic to the constant loop taking the homotopy $h_t = \gamma_t \cdot \gamma_t^-$, where γ_t equals γ in the interval $[0, 1-t]$ and is stationary afterwards. So, restricting that to loops, we have that $\gamma \cdot \gamma^- \sim c \Rightarrow [\gamma] \cdot [\gamma^-] \sim [c]$. □

We will see that if X is path-connected, then the group $\pi_1(X, x_0)$ is independent of the choice of basepoint x_0, up to isomorphism. So, we can talk about the fundamental group of the space X, denoted $\pi_1(X)$.

Proposition 2.6. *If the points $x_0, x_1 \in X$ are path-connected, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. Since X is path-connected, there is a path h from x_0 to x_1. So, taking a loop γ at the basepoint x_0, we can have the loop $h \cdot \gamma \cdot h^- \sim (h \cdot \gamma) \cdot h^- \sim h \cdot (\gamma \cdot h^-)$ at the basepoint x_1. Therefore, we can define the homomorphism $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$, with $[\gamma] \mapsto [h \cdot \gamma \cdot h^-]$, which is well-defined, since if γ_t is a homotopy of loops based at x_0, then $h \cdot \gamma_t \cdot h^-$ is a homotopy of loops based at x_1, and inversely.

We see that β_h is a homomorphism, since $\beta_h([\gamma \cdot \eta]) = [h \cdot \gamma \cdot \eta \cdot h^-] = [h \cdot \gamma \cdot h^- \cdot h \cdot \eta \cdot h^-] = \beta_h([\gamma]) \cdot \beta_h([\eta])$. Furthermore, it is an isomorphism with inverse β_{h^-}, because $\beta_h \circ \beta_{h^-}([\gamma]) = \beta_h([h^- \cdot \gamma \cdot h] = [h \cdot h^- \cdot \gamma \cdot h \cdot h^-] = [\gamma]$, and similarly $\beta_{h^-} \circ \beta_h([\gamma]) = [\gamma]$. □

The fundamental group is a property that characterises a space. A signigicant class of spaces is that of the spaces with trivial fundamental group. We will see that in these spaces all the paths from a point to another are equivalent, concerning homotopy.

Definition 2.7 (**Simply-connected Space**)**.** A path-connected space X is called **simply-connected**, if its fundamental group is trivial.

Proposition 2.8. *A space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X.*

Proof. If γ and η are two paths in the simply-connected space X from x_0 to x_1, then $\gamma \sim \gamma \cdot c_{x_1} \sim \gamma \cdot \eta^- \cdot \eta \sim c_{x_0} \cdot \eta \sim \eta$, since $\eta^- \cdot \eta$ and $\gamma \cdot \eta^-$ are loops at basepoints x_1 and x_0 respectively, so homotopic to constant loops. Conversely, if there is only one homotopy class of paths connecting a basepoint x_0 to itself, then all these loops are homotopic to the constant loop, thus $\pi_1(X, x_0)$ is trivial. □

Finally, we have a definition and a theorem, concerning the fundamental group, which will be useful later.

Definition 2.9. (Induced Homomorphism). Let $\phi : X \to Y$ be a map of topological spaces taking the basepoint x_0 to the basepoint $y_0 \in Y$. Then ϕ induces a homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, defined by composing loops $\gamma : [0, 1] \to X$ based at x_0 with ϕ, which means $\phi([\gamma]) = [\phi \circ \gamma]$.

This induced homomorphism ϕ_* is well defined, because a homotopy of loops based at x_0, γ_t yields a composed homotopy $\phi \circ \gamma_t$ of loops based at y_0, so $\phi_*([\gamma_0]) = [\phi \circ \gamma_0] = [\phi \circ \gamma_1] = \phi_*([\gamma_1])$. Moreover, it is a homomorphism, since $\phi \circ (\gamma \cdot \eta) = (\phi \circ \gamma) \cdot (\phi \circ \eta)$ and both functions have the value $(\phi \circ \gamma)(2s)$ for $s \in [0, 1/2]$ and the value $(\phi \circ \eta)(2s - 1)$ for $s \in [1/2, 1]$.

A basic property of induced homomorphisms is $(\phi \circ \psi)_* = \phi_* \circ \psi_*$. It is easy to prove this, since the composition of maps is associative $((\phi \circ \psi)(\gamma) = \phi(\psi(\gamma))$.

The last theorem concerns the fundamental group of topological manifolds.

Theorem 2.10. *The fundamental group of a topological manifold is countable.*

Proof. Let X be a topological manifold. By Lemma 1.10 in Lee[4], there is a countable collection $\mathcal{B}$ of coordinate balls covering X. For any pair of coordinate balls $B, B' \in \mathcal{B}$ the intersection $B \cap B'$ has at most countably many components, each of which is path-connected. Let $\mathcal{X}$ be a countable set containing a point from each component of $B \cap B'$ for each $B, B' \in \mathcal{B}$ (including $B = B'$). For each $B \in \mathcal{B}$ and each $x, x' \in \mathcal{X}$, such that $x, x' \in B$, let $h^B_{x,x'}$ be some path from x to x' in B.

Since the fundamental groups based at any two points in the same component of X are isomorphic (from proposition 2.6), and $\mathcal{X}$ contains at least one point in each component of X, we may as well choose a point $p \in X$ as basepoint. Define a special loop to be a loop based at p that is equal to a finite product of paths of the form $h^B_{x,x}$. Clearly, the set of special loops is countable, and each special loop determines an element of $\pi_1(X,p)$. Therefore, to show that $\pi_1(X,p)$ is countable, it suffices to show that each element of $\pi_1(X,p)$ is represented by a special loop.

Let $\gamma : [0,1] \to X$ be a loop based at p. The collection of components of sets of the form $\gamma^{-1}(B)$ as B ranges over $\mathcal{B}$ is an open cover of $[0,1]$, so by compactness it has a finite subcover. Thus, there are finitely many numbers $0 = s_0 < s_1 < ... < s_m = 1$ such that $[s_{i-1}, s_i] \subseteq \gamma^{-1}(B)$ for some $B \in \mathcal{B}$. We denote the ball B containing $\gamma([s_{i-1}, s_i])$ by B_i and let γ_i be the path obtained by restricting γ to $[s_{i-1}, s_i]$. Then, it is easy to see that γ is the composition of $\gamma_1, ..., \gamma_m$ with γ_i a path in B_i.

For each i, we have that $\gamma(s_i) \in B_i \cap B_{i+1}$ and there is some $x_i \in \mathcal{X}$ that lies in the same component of $B_i \cap B_{i+1}$ as $\gamma(s_i)$. So, we can choose a path η_i in $B_i \cap B_{i+1}$ from x_i to $\gamma(s_i) \in A_i \cap A_{i+1}$. Hence, we can consider the loop

$$(\gamma_1 \cdot \eta_1^-) \cdot (\eta_1 \cdot \gamma_2 \cdot \eta_2^-) \cdot (\eta_2 \cdot \gamma_3 \cdot \eta_3^-) \cdot ... \cdot (\eta_{m-1} \cdot \gamma_m)$$

which is homotopic to γ, if we erase η_i^- with η_i. But each of the parentheses is a path in B_i from x_{i-1} to x_i and B_i is simply connected, so it is homotopic to $h^{B_i}_{x_{i-1},x_i}$. Therefore, γ is a special loop. ☐

2.3 Example: Calculation of a Fundamental Group

As an example, we will calculate the fundamental group of the sphere S^n for $n \geq 2$. For the case $n = 1$, we need the notion of the covering map, which is introduced at the next section. We need the following lemma:

Lemma 2.11. *If a space X is the union of a collection of path-connected open sets A_a each containing the basepoint $x_0 \in X$ and if each intersection $A_a \cap A_b$ is path-connected, then every loop in X with basepoint x_0 is homotopic to a product of loops each of which is contained in a single A_a.*

Proof. For this proof we use the same technique as in the previous proof in a most general sense. Given a loop $\gamma : [0,1] \to X$ at the basepoint x_0, we claim that there is a partition $0 = s_0 < s_1 < ... < s_m = 1$ such that each subinterval is mapped by γ to a single A_a. We can do that, because γ is continuous, thus each $s \in [0,1]$ has an open neighbourhood $V_s \subset [0,1]$ mapped by γ to A_a. We can select V_s such that its closure is mapped to a single A_a. But $[0,1]$ is compact, so a finite number of these intervals can cover $[0,1]$. The endpoints of these intervals define the desired partition.

We denote the sets A_a containing $\gamma([s_{i-1}, s_i])$ by A_i and let γ_i be the path obtained by restricting γ to $[s_{i-1}, s_i]$. Then, it is easy to see that γ is the composition of $\gamma_1, ..., \gamma_m$ with γ_i a path in A_i. However, we assummed that

$A_i \cap A_{i+1}$ is path-connected, so we can choose a path η_i in $A_i \cap A_{i+1}$ from x_0 to $\gamma(s_i) \in A_i \cap A_{i+1}$. Hence, we can consider the loop

$$(\gamma_1 \cdot \eta_1^-) \cdot (\eta_1 \cdot \gamma_2 \cdot \eta_2^-) \cdot (\eta_2 \cdot \gamma_3 \cdot \eta_3^-) \cdot ... \cdot (\eta_{m-1} \cdot \gamma_m)$$

which is homotopic to γ, if we erase η_i^- with η_i. But each of the parentheses is a loop lying in a single A_i, so the initial loop is homotopic to a product of such loops. □

Theorem 2.12. *The fundamental group of the sphere, $\pi_1(S^n)$, for $n \geq 2$ is trivial.*

Proof. We can take two open sets of the sphere S^n, A_1 and A_2, which are the complements of two antipodal points, each of which are homeomorphic to $\mathbb{R}^n$ by the stereographic projection. Since $n \geq 2$, $A_1 \cap A_2$ is path-connected. So, choosing a basepoint $x_0 \in A_1 \cap A_2$, we apply the previous lemma. Hence, we have that every loop in S^n based at x_0 is homotopic to a product of loops in A_1 and A_2. But $\pi_1(A_1)$ and $\pi_1(A_2)$ are trivial, since A_1 and A_2 are homeomorphic to $\mathbb{R}^n$. Therefore, every loop in S^n is homotopic to the constant loop, so its fundamental group is trivial. □

3 Covering Group

3.1 Covering Space and Covering Manifold

We begin with the definitions of covering space and covering manifold, so we can apply this notion to Lie groups.

Definition 3.1.

- Let X, Y be two path-connected topological spaces. A surjective continuous map $\pi : X \to Y$ is a **covering map** if $\forall y \in Y$, there exists an open neighbourhood U of y in Y, such that $\pi^{-1}(U)$ is a disjoint union of open sets $V_i \subset X$, each of which is mapped homeomorphically onto U by π. Such a neighbourhood is called an **evenly covered neighbourhood**. The sets V_i are called the **sheets** of π. The set $\pi^{-1}(y)$ is called the **fiber** over y.
- A **covering space** of a topological space Y is a topological space X, together with a covering map $\pi : X \to Y$. If X is simply-connected (its fundamental group is trivial), then it is called the **universal covering space** of Y.
- Let X, Y be two connected smooth manifolds. A smooth map $\pi : X \to Y$ is a **smooth covering map**, if it is a topological covering map, but for any $y \in Y$ and a neighbourhood U of y, any open component of $\pi^{-1}(U)$ is mapped diffeomorphically onto U by π.
- If $\pi : X \to Y$ is a smooth covering map, then Y is called the base of the covering and X is called a **covering manifold** of Y. If X is simply-connected, then it is called the **universal covering manifold** of Y.

We have the following properties for smooth covering maps:

1. Every smooth covering map is a local diffeomorphism.
2. Every smooth covering map is a smooth submersion, an open map and a quotient map.
3. Every smooth covering map $\pi : X \to Y$ has constant rank with $rank(\pi_*) = \dim X = \dim Y$. Also, every point of X is regular.
4. An injective smooth covering map is a diffeomorphism.
5. A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.

Proof.

1. For every $x \in X$, we have that for a neighbourhood U of $\pi(x)$, $\pi^{-1}(U)$ is a collection of open sets $\bigcup V_i$, such that $\pi(V_i) = U$ diffeomorphically. But we have that $x \in \pi^{-1}(U)$, so there is always a neighbourhood of x, which is mapped diffeomorphically to Y.
2. Since π is a local diffeomorphism, from Lee 4.8(a)[4] it is also a smooth immersion, a smooth submersion and from Lee 4.28 it is an open map and a quotient map.
3. Since π is a local diffeomorphism, we have that $\forall x \in X$, $\pi_* : T_xX \to T_{\pi(x)}Y$ is an isomorphism, so $rank(\pi_*) = \dim X = \dim Y$.
4. Since a covering map is also surjective and local diffeomorphism, it is a bijective local diffeomorphism, so by Lee 4.6(f) it is a diffeomorphism.
5. Since π is a topological covering map, for all points $x \in X$, there is an open neighbourhood U, such that each connected component of $\pi^{-1}(U)$ is homeomorphic to U. If V is one of these components, then $\pi|_V$ is bijective (from homeomorphism) and local diffeomorphism, thus a diffeomorphism. So, π is a diffeomorphism with each component of $\pi^{-1}(U)$, which means it is a smooth covering map.

□

Another significant property of covering maps is that we can transfer a continuous function from the base of the covering to the covering space, which is called lifting.

3.2 Lifting

Definition 3.2 (Lift). Let $\pi : X \to Y$ be a covering map and $f : Z \to Y$ be a continuous map. Then a **lift** of f is a continuous map $F : Z \to X$ such that $\pi \circ F = f$.

Lemma 3.3 (Unique Path Lifting). *Let $\pi : X \to$ be a covering map, $\gamma : [0,1] \to Y$ a path in Y and x_0 a point in the fiber over $y_0 = \gamma(0)$ ($x_0 \in \pi^{-1}(y_0)$). Then there exists at most one lift of γ, $\Gamma : [0,1] \to X$, such that $\Gamma(0) = x_0$.*

Proof. We prove that by contradiction. Suppose we have two such lifts $\Gamma_1, \Gamma_2 : [0,1] \to X$. We take the set

$$S := \{t \in [0,1] : \Gamma_1(t) = \Gamma_2(t)\}.$$

The set S is non-empty, because $0 \in S$ and it is also closed (all its limit points are in the set, because Γ_1, Γ_2 are continuous).

So, it suffices to prove that it is also open. We will prove that $\forall s \in S$, there exists $\epsilon > 0$, such that $[s-\epsilon, s+\epsilon] \cap [0,1] \subset S$. We will consider only the case $s = 0$ for simplicity, but the general case is entirely similar.

So, we want to prove that there exists $\epsilon > 0$, such that $[0, \epsilon] \subset S$. We pick a small open neighbourhood U of x_0 such that π restricts to a homeomorphism onto $\pi(U)$ (we take the element of $\pi^{-1}(y_0)$ that includes x_0). Hence, there exists an $\epsilon > 0$, such that $\Gamma_i \subset U$ for $i = 1, 2$. However, $\pi \circ \Gamma_1 = \pi \circ \Gamma_2 = \gamma$, so since π is a homeomorphism in U, we take $\Gamma_1|_{[0,\epsilon]} = \Gamma_2|_{[0,\epsilon]}$.

Thus, S is both open and closed, so $S = [0,1] \Rightarrow \Gamma_1 = \Gamma_2$. □

Theorem 3.4 (**Unique Lifting**). *Let $\pi : X \to Y$ be a covering map, and $f : Z \to Y$ be a continuous map, where Z is a connected space. If we fix $z_0 \in Z$ and $x_0 \in \pi^{-1}(y_0)$, where $y_0 = f(z_0)$, then there exists at most one lift $F : Z \to X$ of f such that $F(z_0) = x_0$.*

Proof. Let a_z be a continuous path $\forall z \in Z$ that connects z_0 to z. If F_1, F_2 are two lifts of f such that $F_1(z_0) = F_2(z_0) = x_0$, then $\forall z \in Z$, the paths $\Gamma_1 = F_1(a_z)$ and $\Gamma_2 = F_2(a_z)$ are two lifts of the path $\gamma = f(a_z)$ that start at the same point. So, from the previous lemma, we get that $\Gamma_1 = \Gamma_2 \Rightarrow \Gamma_1(1) = \Gamma_2(1) \Rightarrow F_1(z) = F_2(z), \forall z \in Z$. □

So, we have proved uniqueness of lifts. Next, we prove the existence of lifts for different types of maps.

Theorem 3.5 (**Homotopy lifting property**). *Let $\pi : X \to Y$ be a covering map, $f : Z \to Y$ be a continuous map and $F : Z \to X$ be a lift of f. If*

$$h : [0,1] \times Z \to Y \quad (t, z) \mapsto h_t(z)$$

is a homotopy of f ($h_0(z) \equiv f(z)$), then there exists a unique lift of h:

$$H : [0,1] \times Z \to Z \quad (t, z) \mapsto H_t(z)$$

such that $H_0(z) \equiv F(z)$.

Proof. For each $z \in Z$ we can find an open neighbourhood U_z, and a partition $0 = t_0 < t_1 < ... < t_n = 1$, depending on z, such that h maps $[t_{i-1}, t_i] \times U_z$ into an evenly covered neighbourhood of $h_{t_{i-1}}(z)$. Following this partition, we can now lift $h|_{[0,1]\times U_z}$ to a continuous map $H = H^z : [0,1] \times U_z \to X$ such that $H_0(\zeta) = F(\zeta), \forall \zeta \in U_z$.

By unique lifting property, the liftings on $[0,1] \times U_{z_1}$ and $[0,1] \times U_{z_2}$ must agree on $[0,1] \times (U_{z_1} \cap U_{z_2})$, $\forall z_1, z_2 \in Z$, and therefore we can glue all these local lifts together to obtain the desired lift H. □

Corollary 3.6 (**Path lifting property**). *Let $\pi : X \to Y$ be a covering map, $y_0 \in Y$, and $\gamma : [0,1] \to Y$ is a continuous path starting at y_0. Then, $\forall x_0 \in \pi^{-1}(y_0)$, there exists a unique lift $\Gamma : [0,1] \to X$ of γ starting at x_0.*

Proof. We use the previous theorem with $f : \{pt\} \to Y$, $f(pt) = \gamma(0) = y_0$, its lift $F : \{pt\} \to X$, $F(pt) = \Gamma(0) = x_0$, and $\gamma(t) = h_t(pt)$, thus we take the unique lift $\Gamma(t) = H_t(pt)$. □

Corollary 3.7 (**Monodromy of the covering**). *Let $\pi : X \to Y$ be a covering map, and $y_0 \in Y$. If γ_0 and γ_1 are loops based on y_0, which are homotopic, then any lifts Γ_0, Γ_1, which start at the same point also end at the same point.*

Proof. We lift the homotopy γ_t connecting γ_0 to γ_1 to a homotopy Γ_t in X. By the homotopy lifting property, this lift connects Γ_0 to Γ_1. Thus, we get a continuous path $\Gamma_t(1)$ inside the fiber $\pi^{-1}(y_0)$, which connects $\Gamma_0(1)$ to $\Gamma_1(1)$. But since the fibers are discrete, this path must be constant. □

Theorem 3.8. *Let $\pi : X \to Y$ be a covering map, $x_0 \in X$ and $y_0 = \pi(x_0)$. Then, the induced homomorphism $\pi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is injective. The image subgroup $\pi_*(\pi_1(X, x_0))$ in $\pi_1(Y, y_0)$ consists of the homotopy classes of loops in Y based at y_0 whose lifts to X starting at x_0 are loops.*

Proof. An element of the kernel of π_* is represented by a loop $F_0 : [0,1] \to X$ with a homotopy $f_t : [0,1] \to X$ of $f_0 = \pi \circ F_0$ to the trivial loop f_1. From the path lifting property, every f_t is uniquely lifted to a path $F_t : [0,1] \to X$, which is also a homotopy of paths, since as t varies each endpoint of F_t traces out a path lifting a constant path, therefore it is a constant path. This homotopy has F_0 the initial path and F_1 the unique lift of the constant loop f_1, so a constant loop. Thus, F_0 is homotopic to the constant loop and the kernel of π_* is trivial. As a result, π_* is injective.

For the second result, we see that loops at y_0 lifting to loops at x_0 certainly represent elements of the image of π_*. Conversely, a loop representing an element of the image of π_* is homotopic to a loop having such a lift, so by the homotopy lifting property, the loop itself must have a lift. □

Theorem 3.9 (**Existence of lifts**). *Let $\pi : X \to Y$ be a covering map, $x_0 \in X$, $y_0 = \pi(x_0) \in Y$, $f : Z \to Y$ a continuous map and $z_0 \in Z$ such that $f(z_0) = y_0$. Assuming that the spaces Y and Z are connected, f admits a lift $F : Z \to X$ such that $F(z_0) = x_0$ if and only if*

$$f_*(\pi_1(Z, z_0)) \subset \pi_*(\pi_1(X, x_0)).$$

Proof. "$\Rightarrow$"

If F is such a lift, then differentiating the relation $f = \pi \circ F$, we get $f_*(\pi_1(Z, z_0)) = \pi_* \circ F_*(\pi_1(Z, z_0) \subset \pi_*(\pi_1(X, x_0))$.

"$\Leftarrow$"

For any $z \in Z$, we choose a path γ_z from z_0 to z. Then $a_z = f(\gamma_z)$ is a path from y_0 to $y = f(z)$ (since f is continuous). We denote by A_z the unique lift of a_z starting at x_0, and set $F(z) = A_z(1)$.

We will show that F is a well defined map. Indeed, let γ' be another path from z_0 to z. Then, $f(\gamma') \cdot a_z^-$ is a loop h_0 at y_0. But $h_0 = f(\gamma') \cdot f(\gamma_z)^- = f(\gamma' \cdot \gamma_z)^-$, so $[h_0] \in f_*(\pi_1(Z, z_0)) \subset \pi_*(\pi_1(X, x_0))$. This means that there is a homotopy h_t of h_0 to a loop h_1 that lifts to a loop H_1 in X, based at x_0, with $\pi_*(H_1) = h_1$. Then, from the homotopy lifting property, we get a lifting H_t of h_t. Since H_1 is a loop at x_0, so is H_0. By the uniqueness of lifted paths, we have that the first half of H_0 is the lift of $f(\gamma')$, namely Γ', and the second half is A_z traversed backwards. So, the midpoint is common, therefore $A_z(1) = \Gamma(1) = F(z)$. This shows that F is well defined.

It suffices to show that F is continuous. Let $U \subset Y$ be an open neighbourhood of $f(z)$ having a lift $\tilde{U} \subset X$ containing $F(z)$ such that $\pi : \tilde{U} \to U$ is a homeomorphism. We choose a path connected open neighbourhood V of z with $f(V) \subset U$. For paths from z_0 to points $z' \in V$, we take a fixed path γ from z_0 to z followed by a path η in V from z to z'. Then, the paths $f(\gamma)$, $f(\eta)$ in Y have lifts $F(\gamma), F(\eta)$ in X, and $f(\gamma) \cdot f(\eta)$ has lift $F(\gamma) \cdot H$, where $H = \pi^{-1}(f(\eta))$ and comes from the inverse of the homeomorphism $\pi : \tilde{U} \to U$. Thus $F(z') = F(\eta(1)) = H(1) \in \pi^{-1}(f(\eta(1))) = \pi^{-1}(z')$, which means that $F(V) \subset \tilde{U}$ and $F|_V = \pi^{-1} \circ f$, hence F is continuous at z. □

Finally, using the notion of lifts, we can define a map from the fundamental group of a point to its fiber in a covering.

Definition 3.10 (**Lifting Correspondence**)**.** Let $\pi : X \to Y$ be a covering map, $x_0 \in X$ a basepoint and $y_0 = \pi(x_0) \in Y$. Given an element $[\gamma]$ of $\pi_1(Y, y_0)$, let Γ be the lifting of γ to a path in X that begins at x_0. Let the map

$$\phi : \pi_1(Y, y_0) \to \pi^{-1}(y_0)$$

be such that $\phi([\gamma])$ denotes the endpoint $\Gamma(1)$. We call ϕ the **lifting correspondence** derived from the covering map π.

From Theorem 3.7, ϕ is well-defined. We have also a very important result for the lifting correspondence.

Proposition 3.11. *Let $\pi : X \to Y$ be a covering map, $x_0 \in X$ a basepoint and $y_0 = \pi(x_0) \in Y$. If X is path-connected, then the lifting correspondence $\phi : \pi_1(Y, y_0) \to \pi^{-1}(y_0)$ is surjective. If X is simply connected, it is bijective.*

Proof. Let $x \in \pi^{-1}(y_0)$. Then there is a path Γ in X from x_0 to x. So, $\gamma = \pi \circ \Gamma$ is a loop in Y at basepoint y_0, so $\phi([\gamma]) = x$. Thus, ϕ is surjective.

Suppose X is simply connected. Let $[\gamma], [\eta]$ be two elements of $\pi_1(Y, y_0)$ such that $\phi([\gamma]) = \phi([\eta])$. We take Γ and H to be their liftings in X respectively starting at point x_0 and ending at the same point $x = \phi([\gamma]) = \phi([\eta])$. X is simply connected, so there is a homotopy Γ_t between Γ and H, by proposition 2.8. Therefore, $\pi \circ \Gamma_t$ is a homotopy between γ and η, which means that $[\gamma] = [\eta]$. So, ϕ is also injective, thus bijective. □

3.3 Application in Fundamental Group

The idea of a covering space is very useful for calculating the fundamental groups of various spaces. For example in the final chapter, we will calculate the fundamental group of the Lie group $SO(3)$. We have already calculated the fundamental group for the sphere S^n, if $n \geq 2$. Now, we have the necessary tools to calculate the fundamental group of the circle S^1.

Theorem 3.12. $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. We construct a covering map of $\pi : \mathbb{R} \to S^1$ with $s \mapsto (\cos(2\pi s), \sin(2\pi s))$. This map is surjective and for any open neighbourhood U, $\pi^{-1}(U)$ is a union of disjoint open sets in $\mathbb{R}$ (with period 1).

So, let $\gamma : [0,1] \to S^1$ be a loop at the basepoint $x_0 = (1,0)$, representing a given element of $\pi_1(S^1, x_0)$. By the path lifting property, there is a lift Γ of γ that starts at 0 and ends at some integer n, since $\pi(\Gamma(1)) = \gamma(1) = x_0$ and $\pi^{-1}(x_0) = \mathbb{Z}$. Therefore, since the path lifting is unique, we get a homomorphism $h : \pi_1(S^1, x_0) \to \mathbb{Z}$, which is well-defined.

To prove that it is surjective, we construct the loops in S^1, $\omega_n : [0,1] \to S^1$ with $\omega_n(s) = (\cos(2\pi ns)), \sin(2\pi ns)$, which are uniquely lifted to the paths $\Omega_n(s) = ns$, since $\pi \circ \Omega_n = \omega_n$. However, Ω_n is a path from 0 to an arbitrary integer n, so we have surjectivity.

It suffices to show that it is injective. We have seen that for any loop γ, there is a unique lifting path that goes from 0 to n for some n. But also, ω_n is lifted to a path from 0 to n. So, we have that $\Gamma \sim \Omega_n$ by the homotopy $(1-t)\Gamma + t\Omega_n$, which if it is composed with π gives a homotopy for $\gamma \sim \omega_n$. Therefore, if $h([\gamma_1]) = h([\gamma_2]) = n$, then $\gamma_1 \sim \omega_n \sim \gamma_2 \Rightarrow [\gamma_1] = [\gamma_2]$. □

3.4 Universal Covering

We have already defined universal covering. In this subsection, we will prove the the existence of it in the case of spaces and manifolds.

Theorem 3.13 (Existence of Universal Covering Map). *If X is a connected and locally simply connected topological space, there exists a simply connected topological space $\tilde{X}$ and a covering map $\pi : \tilde{X} \to X$.*

Remark. The universal covering space is unique in the following sense: if $\tilde{X}'$ is any other simply connected space that admits a covering map $\pi' : \tilde{X}' \to X$, then there exists a homeomorphism $\Phi : \tilde{X} \to \tilde{X}'$ such that $\pi' \circ \Phi = \pi$. However, we will not prove this result.

Proof. We construct such a covering space. Given a basepoint x_0, we define

$$\tilde{X} = \{[\gamma] | \ \gamma : [0,1] \to X, \gamma(0) = x_0\}$$

where $[\gamma]$ denotes the homotopy classes γ with respect to homotopies that fix endpoints $\gamma(0)$ and $\gamma(1)$. Then, the map $\pi : \tilde{X} \to X$ with $[\gamma] \mapsto \gamma(1)$ is well-defined, and since X is path-connected, the endpoint $\gamma(1)$ can be any point of X, so π is surjective.

Topology of $\tilde{X}$

Let $\mathcal{U}$ be the collection of path-connected open sets $U \subset X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial (such U is simply connected and exists, because X is locally simply connected). Given a set $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \ \eta : [0,1] \to U, \eta(0) = \gamma(1)\}.$$

$U_{[\gamma]}$ depends only on the homotopy class $[\gamma]$. We see that $\pi : U_{[\gamma]} \to U$ is surjective, since U is path-connected, and injective, since different choices of η joining $\gamma(1)$ to a fixed $x \in U$ are all homotopic in X from the trivial map $\pi_1(U) \to \pi_1(X)$.

Another property of such sets is that $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$, because if $\gamma' = \gamma \cdot \eta$, then the elements of $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$, so they lie in $U_{[\gamma']}$, while the elements of $U_{[\gamma]}$ have the form $[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \eta^{-1} \cdot \mu] = [\gamma' \cdot \eta^{-1} \cdot \mu]$, so they lie in $U_{[\gamma']}$.

We use the last property to show that the sets $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$, since if we take two such sets $U_{[\gamma]}, V_{[\eta]}$ and an element at its intersection $[\mu] \in U_{[\gamma]} \cap V_{[\eta]}$, then we have $U_{[\gamma]} = U_{[\mu]}$ and $V_{[\eta]} = V_{[\mu]}$. So, if $W \in \mathcal{U}$, $W \subset U \cap V$ and contains $\mu(1)$, then $W_{[\mu]} \subset U_{[\mu]} \cap V_{[\mu]}$ and $[\mu] \in W_{[\mu]}$.

$\tilde{X}$ is a covering space of X

The bijection $\pi : U_{[\gamma]} \to U$ is a homeomorphism, since it is a bijection between the subsets $V_{[\eta]} \subset U_{[\gamma]}$ and the sets $V \in \mathcal{U}$ contained in U. To verify this, in one direction we have $\pi(V_{[\eta]}) = V$ and in the other direction we have $\pi^{-1}(V) \cap U_{[\gamma]} = V_{[\eta]}$ for any $[\eta] \in U_{[\gamma]}$ with endpoint in V, because $V_{[\eta]} \subset U_{[\eta]} = U_{[\gamma]}$ and $V_{[\eta]}$ maps onto V.

We have shown that $\pi : \tilde{X} \to X$ is a local homeomorphism. To complete the proof that it is a covering map, we have that for a fixed $U \in \mathcal{U}$, the sets $U_{[\gamma]}$ for varying $[\gamma]$, partition $\pi^{-1}(U)$, because if $[\mu] \in U_{[\gamma]} \cap U_{[\gamma']}$, then $U_{[\gamma]} = U_{[\mu]} = U_{[\gamma']}$.

$\tilde{X}$ is path-connected

For a point $[\gamma] \in \tilde{X}$, let γ_t be the path in X that equals γ on $[0,t]$ and is stationary at $\gamma(t)$ on $[t,1]$. Then the function $t \mapsto [\gamma_t]$ is a path in $\tilde{X}$ lifting γ that starts at $[x_0]$, the homotopy class of the constant path at x_0, and ends at $[\gamma]$. Since $[\gamma]$ is an arbitrary point in $\tilde{X}$, $\tilde{X}$ is path connected.

$\tilde{X}$ is simply-connected

It suffices to show that $\pi_1(\tilde{X}, [x_0])$ is trivial or equivalently that its image under π_* is trivial, since π_* is injective (by Theorem 3.8). The elements in the image of π_* are represented by loops γ at x_0 that lift to loops in $\tilde{X}$ at $[x_0]$. As we saw in the previous paragraph the path $t \mapsto [\gamma_t]$ lifts γ starting at $[x_0]$ and, if this lifted path is a loop, this means that $[\gamma_1] = [x_0]$. But $\gamma_1 = \gamma$, thus $[\gamma] = [x_0]$, so γ is homotopic with the trivial loop. As a result the image of π_* is the trivial loop. □

In the case of manifolds:

Theorem 3.14. *Let $\pi : X \to Y$ be a topological covering map and suppose Y is a connected smooth n-manifold. Then, X is also a topological n-manifold, and has a smooth structure such that π is a smooth covering map.*

Proof. Since π is a local homeomorphism, X is locally Euclidean. To show that it is Hausdorff, let x_1 and x_2 be distinct points in X. If $\pi(x_1) = \pi(x_2)$ and $U \subseteq Y$ is an evenly covered open subset containing $\pi(x_1)$, then the components of $\pi^{-1}(U)$ containing x_1 and x_2 are dijoint open subsets of X that separate x_1 and x_2. On the other hand, if $\pi(x_1) \neq \pi(x_2)$, there are disjoint open subsets $U_1, U_2 \subseteq Y$ containing $\pi(x_1)$ and $\pi(x_2)$, respectively, and then $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint open subsets of X containing x_1 and x_2. Thus, X is Hausdorff.

To show that X is second-countable (has a countable base), we will first show that each fiber of π is countable. Given $y \in Y$ and an arbitrary point $x_0 \in \pi^{-1}(y)$, we have the lifting correspondence $\phi : \pi_1(Y, y) \to \pi^{-1}(y)$, which is surjective by Proposition 3.11. This shows second-countability, since from Theorem 2.10 every fundamental group of a manifold is countable.

The collection of all evenly covered subsets is an open cover of Y, and therefore has a countable subcover $\{U_i\}$. For any given i, each component of $\pi^{-1}(U_i)$ contains exactly one point in each fiber of the points of U_i, so $\pi^{-1}(U_i)$ has countably many components. Hence, the collection of all components of all sets of the form $\pi^{-1}(U_i)$ is a countable open cover of X. Since each such component is second-countable, then X is second countable. This completes the proof that X is a topological manifold.

To construct a smooth structure on X, suppose x is any point in X, and let U be an evenly covered neighbourhood of $\pi(x)$. After shrinking U if necessary, we may assume also that it is the domain of a smooth coordinate map $\phi : U \to \mathbb{R}^n$. If $\tilde{U}$ is the component of $\pi^{-1}(U)$ containing x, and $\tilde{\phi} = \phi \circ \pi|_{\tilde{U}} : \tilde{U} \to \mathbb{R}^n$, then $(\tilde{U}, \tilde{\phi})$ is a chart on X. If two such charts $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ overlap, the transition map can be written

$$\tilde{\psi} \circ \tilde{\phi}^{-1} = (\psi \circ \pi|_{\tilde{U} \cap \tilde{V}}) \circ (\phi \circ \pi|_{\tilde{U} \cap \tilde{V}})^{-1} = \psi \circ (\pi|_{\tilde{U} \cap \tilde{V}}) \circ (\pi|_{\tilde{U} \cap \tilde{V}})^{-1} \circ \phi^{-1} = \psi \circ \phi^{-1},$$

which is smooth. Thus the collection of all such charts defines a smooth structure on X.

Finally, π is a smooth covering map, because its coordinate representation in terms of pair of charts $(\tilde{U}, \tilde{\phi})$ and (U, ϕ) is the identity. ☐

Corollary 3.15 (Existence of a Universal Covering Manifold). *If X is a connected smooth manifold, there exists a simply connected smooth manifold $\tilde{X}$, called the universal covering manifold of X, and a smooth covering map $\pi : \tilde{X} \to X$.*

Remark. The universal covering manifold is unique in the following sense: if $\tilde{X}'$ is any other simply connected smooth manifold that admits a smooth covering map $\pi' : \tilde{X}' \to X$, then there exists a diffeomorphism $\Phi : \tilde{X} \to \tilde{X}'$ such that $\pi' \circ \Phi = \pi$. However, we will not prove this result.

Proof. This is a result of the existence of the universal topological cover and the theorem, which asserts that this topological space has a smooth manifold structure. □

3.5 Universal Covering in Lie Groups

Covering is a topological property, so we define the universal covering group of a Lie group as the universal covering manifold of the manifold structure of the Lie group. Now, we prove that the covering manifold is also a group.

Theorem 3.16 (Existence of a Universal Covering Group). *Let G be a connected Lie group. Then there exists a simply connected Lie group $\tilde{G}$, called the universal covering group of G, that admits a smooth covering map $\pi : \tilde{G} \to G$ that is also a Lie group homomorphism.*

Proof. Let $\tilde{G}$ be the universal covering manifold of G (by Corollary 3.15) and $\pi : \tilde{G} \to G$ be the corresponding smooth covering map. Then $\pi \times \pi : \tilde{G} \times \tilde{G} \to G \times G$ is also a smooth covering.

To see the last fact, we take a point $(g_1, g_2) \in \tilde{G} \times \tilde{G}$ and evenly covered U_1 and U_2 of g_1 and g_2 respectively. Then, since $(\pi \times \pi)^{-1}(U_1 \times U_2) = \pi^{-1}(U_1) \times \pi^{-1}(U_2)$, the components of $(\pi \times \pi)^{-1}(U_1 \times U_2)$ are of the form $V_1^i \times V_2^j$, where V_1^i is a component of $\pi^{-1}(U_1)$ and V_2^j is a component of $\pi^{-1}(U_2)$. As a result, $(\pi \times \pi)(V_1^i \times V_2^j) = U_1 \times U_2$ diffeomorphically.

We define the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ of G. Let $\tilde{e}$ be an arbitrary element of the fiber $\pi^{-1}(e) \subseteq \tilde{G}$. Since $\tilde{G}$ is simply connected, we can use the lifting property of covering maps (Theorem 3.9). Thus, the map $m \circ (\pi \times \pi) : \tilde{G} \times \tilde{G} \to G$ has a unique continuous lift $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ satisfying $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$ and $\pi \circ \tilde{m} = m \circ (\pi \times \pi)$.

Because π is local diffeomorphism, we can deduce from the last relation that $\tilde{m}$ is smooth.

With the same reasoning the map $i \circ \pi : \tilde{G} \to G$ has a smooth lift $\tilde{i} : \tilde{G} \to \tilde{G}$, satisfying $\tilde{i}(\tilde{e}) = \tilde{e}$ and $\pi \circ \tilde{i} = i \circ \pi$.

Hence, we can define multiplication and inversion in $\tilde{G}$ by $xy = \tilde{m}(x, y)$ and $x^{-1} = \tilde{i}(x)$, $\forall x, y \in \tilde{G}$. As a result, we have that $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^{-1}) = \pi(x)^{-1}$ from the relations $\pi \circ \tilde{m} = m \circ (\pi \times \pi)$ and $\pi \circ \tilde{i} = i \circ \pi$.

If we show that $\tilde{G}$ is a group with these operations, then the above show that π is a homomorphism.

First, we show that $\tilde{e}$ is an identity for multiplication in $\tilde{G}$. We consider the map $f : \tilde{G} \to \tilde{G}$ defined by $f(x) = \tilde{e}x$. Then $\pi \circ f(x) = \pi(\tilde{e})\pi(x) = e\pi(x) = \pi(x)$. We conclude that f is a lift of $\pi : \tilde{G} \to G$. The identity map $Id_{\tilde{G}}$ is another lift of π, which agrees with f at a point, since $f(\tilde{e}) = \tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$, so by uniqueness of lifts (Theorem 3.4), we get $f = Id_{\tilde{G}} \Rightarrow \tilde{e}x = x$, $\forall x \in \tilde{G}$. Similarly, $x\tilde{e} = x$.

Next, we show that multiplication in $\tilde{G}$ is associative. We consider the two maps $a_L, a_R : \tilde{G} \times \tilde{G} \times \tilde{G} \to \tilde{G}$ defined by $a_L(x, y, z) = (xy)z$ and $a_R(x, y, z) =$

$x(yz)$. Then we have:

$$\pi \circ a_L(x, y, z) = (\pi(x)\pi(y))\pi(z) = \pi(x)(\pi(y)\pi(z)) = \pi \circ a_R(x, y, z).$$

Thus, a_L and a_R are both lifts of the same map and they agree at $(\tilde{e}, \tilde{e}, \tilde{e})$, so they are equal. Similarly, we can show that $x^{-1}x = xx^{-1} = \tilde{e}$, so $\tilde{G}$ is a group. $\square$

Remark. In the above proof, we have not used that the manifold $\tilde{G}$ is simply-connected, so we can conclude that any covering manifold of a Lie group is also a Lie group.

Corollary 3.17. *For any connected Lie group G, the universal covering group is unique in the following sense: if $\tilde{G}$ and $\tilde{G}'$ are simply connected Lie groups that admit smooth covering maps $\pi : \tilde{G} \to G$ and $\pi' : \tilde{G}' \to G$ that are also Lie group homomorphisms, then there exists a Lie group isomorphism $\Phi : \tilde{G} \to \tilde{G}'$ such that $\pi' \circ \Phi = \pi$.*

Proof. This is a result of the uniqueness of covering manifold and the fact that the covering manifold of a Lie group can have the structure of a Lie group. $\square$

Theorem 3.18. *Let G be a Lie group and $\tilde{G}$ a covering group of it, with the smooth covering map $\pi : \tilde{G} \to G$. Then the Lie algebras of G and $\tilde{G}$ are isomorphic.*

Proof. We have that $Lie(G) \cong T_eG$. Since the smooth covering map $\pi : \tilde{G} \to G$ is regular, the map $\pi_* : T_{\tilde{e}}\tilde{G} \to T_eG$ is surjective. Using the fact that $\dim \tilde{G} = \dim G$, we conclude that $T_{\tilde{e}}\tilde{G} \cong T_eG$, or equivalently $Lie(\tilde{G}) \cong Lie(G)$. $\square$

Corollary 3.19. *If two Lie groups G_1 and G_2 have the same universal covering up to isomorphism, then they have isomorphic Lie algebras.*

We have seen that Lie groups that have the same universal covering group, have also the same Lie algebra. We can also prove the reverse: if two Lie groups share the same Lie algebra, then they have the same universal covering group. This results from the one-to-one correspondence between simply connected Lie groups and finite-dimensional Lie algebras. This correspondence is called Lie Correspondence.

4 Lie correspondence

Definition 4.1 (**Lie Subgroup and Lie Subalgebra**)**.** A **Lie subgroup** H of a Lie group G is called Lie subgroup if it is a Lie group (with respect to the induced group operation), and the inclusion map $i_H : H \hookrightarrow G$ is a smooth immersion (and therefore a Lie group homomorphism).

A **Lie subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ which is closed under the Lie bracket.

Theorem 4.2. *If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$.*

Proof. Let $X_1, X_2, ..., X_k$ be a basis of $\mathfrak{h} \subset \mathfrak{g}$. From the definition of Lie algebra, the vector fields X_i are left invariant and linearly independent at the identity e. As a result, they are linearly independent at all $g \in G$ $(X_g = L_g(X_e))$. In other words, we have:

$$V_g = span\{(X_1)_g, ..., (X_k)_g\}$$

which is a k-dimensional distribution on G, which is also involutive from the properties of Lie algebras. Thus, by Frobenius theorem there is a unique maximal integral manifold of V through e. We denote this by H.

We need to show that H is a subgroup of G. Let $h_1, h_2 \in H$. We have $h_1 = L_{h_1}e \in H \cap L_{h_1}H$. Since H is an integral manifold, then $L_{h_1}H$ is also an integral manifold. Also, H is maximal and its intersection with $L_{h_1}H$ is not empty, hence $L_{h_1}H \subset H$. Then, we get $h_1h_2 = L_{h_1}(h_2) \in H$. Similarly, $L_{h_1^{-1}}(h_1) = e \in H$, so $L_{h_1^{-1}}H \subset H \Rightarrow L_{h_1^{-1}}(e) = h_1^{-1} \in H$. So, H is a subgroup of G. The group operations are the same as in G, so they are smooth. As a result, we get that H is a Lie group.

For uniqueness, let K be another connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then K is also an integral manifold of V, getting $K \subset H$ (H is maximal). Since $T_eK = T_eH$, there is a neighbourhood of e that the inclusion is isomorphism. However, we can generate any element of a connected Lie group by any open set containing e (see proof of Theorem 5.4), so this local isomorphism extends to the isomorphism $K \cong H$. □

Lemma 4.3. *Suppose G and H are connected Lie groups and $\Phi : G \to H$ is a Lie group homomorphism. If $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ is isomorphism, then Φ is a covering map.*

Proof. Since Φ_* is an isomorphism between T_eG and T_eH, Φ is a local diffeomorphism in a neighbourhood of $e \in H$. Since, H is connected, we can extend Φ to all elements of H (see proof of Theorem 5.4), so Φ is surjective.

It suffices to check the covering property. It is sufficient to check only for $e \in H$ due to group invariance. Again, by the isomorphism Φ_*, Φ maps a neghbourhood U of $e \in G$ bijectively to a neghbourhood V of $e \in H$. Let $\Gamma = \Phi^{-1}(e) \subset G$. Then Γ is a subgroup of G and for any $a \in \Gamma$,

$$\Phi \circ L_a(g) = \Phi(ag) = \Phi(a)\Phi(g) = \Phi(g).$$

So, $\Phi^{-1}(V) = \bigcup_{a \in \Gamma} L_aU$. Then, we are done if we show that $L_{a_1}U \cap L_{a_2}U = \emptyset$ for $a_1 \neq a_2 \in \Gamma$. We will show this by contradiction. Let $a = a_1^{-1}a_2$. If $L_{a_1}U \cap L_{a_2}U \neq \emptyset$, then $L_aU \cap U \neq \emptyset$. We take $p_2 = ap_1 \in L_aU \cap U$, where $p_1, p_2 \in U$. Then $\Phi(p_2) = \Phi(ap_1) = \Phi(p_1)$. However, Φ is injective on U, so $p_1 = p_2 \Rightarrow a = e \Rightarrow a_1 = a_2$, contradiction. □

Corollary 4.4. *Let $\Phi : G \to H$ be a Lie group homomorphism with $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ being isomorphism. If G is connected and H is simply-connected, then Φ is a Lie group isomorphism.*

Proof. From previous Lemma, Φ is a covering map. Since H is simply connected, we get that Φ is a homeomorphism. Thus, Φ and Φ^{-1} are both continuous Lie group homomorphisms, which means that they are smooth. As a result, Φ is a diffeomorphism. □

Theorem 4.5. *Suppose G and H are Lie groups with G simply-connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Then for any Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, there is a unique Lie group homomorphism $\Phi : G \to H$, such that $\Phi_* = \phi$.*

Proof. Suppose $\mathfrak{k} = graph(\phi) = \{(X, \phi(X)) : X \in \mathfrak{g}\}$. It is easy to see that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g} \times \mathfrak{h}$. By the Theorem 4.2, there exists a unique subgroup K of $G \times H$ with $\mathfrak{k}$ as its Lie algebra.

So, considering the inclusion $i : K \to G \times H$ and the projection $pr_1 : G \times H \to G$, we define the Lie group homomorphism $\Psi : K \to G$, $\Psi = pr_1 \circ i$, with $\Psi_* = (pr_1)_* \circ i_*$ being a Lie algebra homomorphism (from $Lie(G) \cong T_eG$). But $\Psi_* : \mathfrak{k} \to \mathfrak{g}$ with $(X, \phi(X)) \mapsto X$ is obviously a bijection. Hence, by the previous corollary, Ψ is a Lie group isomorphism.

Then, we can define $\Phi : G \to H$ as the composition $G \xrightarrow{\Psi^{-1}} K \xrightarrow{pr_2} H$, which has differential $\Phi_* = (\Psi^{-1})_* \circ (pr_2)_* = \phi$.

For the uniqueness, suppose we have two Lie group homomorphisms $\Phi, \Psi : G \to H$, such that $\Phi_* = \Psi_*$. Then, we see that the two graph subalgebras $graph(\Phi_*)$ and $graph(\Psi_*)$ are equal. But again from the Theorem 4.2, there is a unique Lie subgroup of $G \times H$ for this subalgebra. Thus, $graph(\Phi) = graph(\Psi) \Rightarrow \Phi(g) = \Psi(g), \forall g \in G$. □

Corollary 4.6. *If G and H are simply connected Lie groups with isomorphic Lie algebras, then G and H are isomorphic.*

Proof. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G and H respectively. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is the Lie algebra isomorphism, we have from the previous theorem that there are Lie group homomorphisms $\Phi : G \to H$ and $\Psi : H \to G$ such that $\Phi_* = \phi$ and $\Psi_* = \phi^{-1}$. However, both the identity map of G and the composition $\Psi \circ \Phi$ are Lie group homomorphisms from G to itself and their induced Lie algebra homomorphisms are equal to the identity. So, by the uniqueness of the previous theorem, we get $\Psi \circ \Phi = Id_G$. Similarly, $\Phi \circ \Psi = Id_H$, so Φ is a Lie group isomorphism. □

We now have the results necessary for one side of the correspondence. For the other side, we will need Ado's Theorem, which we only state, since its proof is not related with the topic of this project.

Theorem 4.7. ***(Ado's Theorem).*** *Every finite-dimensional real Lie algebra is isomorphic to a Lie subalgebra of some matrix algebra $\mathfrak{gl}(n, \mathbb{R})$ with the commutator bracket.*

Finally, we can prove the theorem of correspondence between Lie algebras and simply-connected Lie groups.

Theorem 4.8 (**The Lie Correspondence**). *There is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups, given by associating each simply connected Lie group with its Lie algebra.*

Proof. We need to show that this correspondence is bijective. If two simply connected Lie groups have isomorphic Lie algebras, then they are isomorphic by Corollary 4.6, so the correspondence is injective. By the previous theorem, each finite-dimensional Lie algebra is isomorphic to a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n, \mathbb{R})$, which by Theorem 4.2 has a unique connected Lie subgroup G of $GL(n, \mathbb{R})$. Taking the universal covering group of G, it has isomorphic Lie algebra to the initial one, so the correspondence is surjective. $\square$

5 Classification of Lie Groups

At this point, we have proved that Lie groups with isomorphic Lie algebras have also isomorphic universal covering groups, which correspond bijectively with finite-dimensional Lie algebras. Now, we will see that from this universal covering group, we can produce all the different Lie groups, up to isomorphism, that have the same Lie algebra. Next, we present some useful results.

Proposition 5.1. *If G is a connected Lie group and $H \subseteq G$ is a discrete subgroup, then the natural quotient map $\pi : G \to G/\Gamma$ is a smooth covering map.*

Proof. The map π is obviously surjective. Let U be an open neighbourhood of e in G so small that $U \cap H = \{e\}$. Let V be a connected open symmetric neighbourhood of e in G such that $V^2 \subseteq U$. We form the open connected sets Vh in G for $h \in H$. If $h_1, h_2 \in H$ such that $Vh_1 \cap Vh_2 \neq \emptyset$, then there exist $v_1, v_2 \in V$, such that $v_1 h_1 = v_2 h_2$, thus $h_1 h_2^{-1} = v_1^{-1} v_2 \in V^2 \cap H \subseteq U \cap H = \{e\}$. Therefore, $h_1 = h_2$ and the sets Vh_i are pairwise disjoint.

Next, let $g_1, g_2 \in Vh$ for some fixed h. If $\pi(g_1) = \pi(g_2)$, then we have $g_1 H = g_2 H$ and hence $g_1 g_2^{-1} \in H$. But writing $g_1 = v_1 h$, $g_2 = v_2 h$ for some $v_1, v_2 \in V$, we get that $v_1 v_2^{-1} = v_1 h \cdot h^{-1} v_2^{-1} = g_1 g_2^{-1} \in H$, thus $v_1 = v_2$ and $g_1 = g_2$. So, π is injective on Vh, and therefore a homeomorphism of Vh onto the open neighbourhood VH of identity in G/H. Translating by each $g \in G$, we get that π is a homeomorphism of each open connected set gVh onto gVH in G/H, and hence π is a covering map as claimed. $\square$

Definition 5.2 (**Normal Lie subgroups**). A Lie subgroup H of G is called **normal subgroup**, if $\forall g \in G$, $gH = Hg$.

Theorem 5.3 (**First Isomorphism Theorem for Lie Groups**). *If $\Phi : G \to H$ is a Lie group homomorphism, then the kernel of Φ is a normal Lie subgroup* *of G, the image of Φ has a unique smooth manifold structure making it into a Lie subgroup of H, and Φ descends to a Lie group isomorphism $\tilde{\Phi} : G/\ker\Phi \to Im\Phi$. If Φ is surjective, then $G/\ker\Phi$ is smoothly isomorphic to H.*

Proof. To begin with, $\ker\Phi$ is a normal subgroup of G, because $\Phi(g\ker\Phi g^{-1}) = \Phi(g)e\Phi^{-1}(g) = e \Rightarrow g\ker\Phi g^{-1} = \ker\Phi$. Also, from group theory, we have that $Im\Phi$ is subgroup of H and Φ descends to a group isomorphism $\tilde{\Phi} : G/\ker\Phi \to Im\Phi$.

However, it is easy to see that $G/\ker\Phi$ also has the structure of manifold, thus it is a Lie group, and the projection $\pi : G \to G/\ker\Phi$ is surjective and has constant rank (Theorem 1.3). Therefore, we get that π is smooth and considering that Φ is also smooth (as a Lie group homomorphism), then from $\Phi = \tilde{\Phi} \circ \pi$ we have that $\tilde{\Phi}$ is also smooth.

Next, the smooth Lie monomorphism $\tilde{\Phi} : G/\ker\Phi \to H$ has constant rank, so it is a smooth immersion. As a result, $Im\Phi$ has also the structure of submanifold, so it is a Lie subgroup. Finally, we have proven that $\tilde{\Phi} : G/\ker\Phi \to Im\Phi$ is a Lie group isomorphism. □

Theorem 5.4. *Let G and H be connected Lie groups. For any Lie group homomorphism $\Phi : G \to H$, the following are equivalent:*

1. *Φ is surjective and has discrete kernel.*
2. *Φ is a smooth covering map.*
3. *Φ is a local diffeomorphism.*
4. *The induced homomorphism $\Phi_* : Lie(G) \to Lie(H)$ is an isomorphism.*

Proof. (1) $\Rightarrow$ (2) Φ is surjective with discrete kernel $\Gamma \subseteq G$. Hence, from Proposition 5.1, the quotient map $\pi : G \to G/\Gamma$ is a smooth covering map and from First Isomorphism Theorem, $\tilde{\Phi} : G/\Gamma \to H$ is a Lie group isomorphism ($Im\Phi = H$, because Φ is surjective). As a result, $\Phi = \tilde{\Phi} \circ \pi$ is itself a smooth covering map.

(2) $\Rightarrow$ (3) From the properties of smooth covering maps, as proven above.

(3) $\Rightarrow$ (1) Since Φ is a local diffeomorphism, it is also a smooth submersion. Also, from the properties of local diffeomorphisms (see the proof of properties for smooth covering maps) $\dim G = \dim H$. Therefore, every set $\Phi^{-1}(h)$, $\forall h \in Im\Phi$ is a submanifold of G of co-dimension equal with $\dim H$, so dimension 0. Hence, $\ker\Phi = \Phi^{-1}(e)$ has dimension 0, so it is a discrete set.

To prove surjectivity, we take a neighbourhood U of $e \in G$, which is mapped diffeomorphically to V, a neighbourhood of $e \in H$. Then, since H is connected, $\forall h \in H$, there is a path $\gamma : [0,1] \to H$ from e to h. We take a partition of $[0,1]$, $0 = s_0 < s_1 < ... < s_m = 1$, such that $\gamma(s_{i_1})^{-1}\gamma(s_i) \in V$ for $i = 1, ..., m$. We can do that because multiplication is smooth and $\gamma(s_i)^{-1}\gamma(s_i) = e$. So, we have that $h = \gamma(s_0)(\gamma(s_0)^{-1}\gamma(s_1))(\gamma(s_1)^{-1}\gamma(s_2))...(\gamma(s_{m-1})^{-1}\gamma(s_m))$, because $\gamma(s_0) = e$ and $\gamma(s_m) = h$. Every element in the parentheses $\gamma(s_{i_1})^{-1}\gamma(s_i)$ is an element of V, so there is some element $g_i \in U$ such that $\Phi(x_i) = \gamma(s_{i_1})^{-1}\gamma(s_i)$. Therefore

$$h = \Phi(x_1)\Phi(x_2)...\Phi(x_m) = \Phi(x_1 x_2 ... x_m)$$

So, Φ is surjective.

(3) $\Rightarrow$ (4) Since Φ is a local diffeomorphism, it is also a smooth immersion and smooth submersion. Thus, $T_eG \cong T_eH \Rightarrow Lie(G) \cong Lie(H)$.

(4) $\Rightarrow$ (3) Since Φ_* is an isomorphism of Lie algebras, we have that Φ is a local diffeomorphism in a neighbourhood of $e \in G$. However, Lie group homomorphisms have constant rank everywhere, so from the local diffeomorphism of the neighbourhood of e, we get $rank(\Phi) = \dim G = \dim H$, therofore Φ is a local diffeomorphism everywhere in G. □

Now, we can prove the main result of the section.

Theorem 5.5 (**Classification of Lie Groups**)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The connected Lie groups whose Lie algebras are isomorphic to $\mathfrak{g}$ are (up to isomorphism) precisely those of the form G_0/D, where G_0 is the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$, and D is a discrete normal subgroup of G.*

Proof. From the Lie Correspondence Theorem (4.8), for any Lie algebra $\mathfrak{g}$ there exists a simply connected Lie group G_0 with Lie algebra isomorphic to $\mathfrak{g}$.

Let H be any other connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$ and $\phi : Lie(G_0) \to Lie(H)$ be the Lie algebra isomorphism. Then, from Theorem 4.5, there is a Lie group homomorphism $\Phi : G_0 \to H$ such that $\Phi_* = \phi$. Also from Theorem 5.4, Φ is a covering map of H and it is surjective with kernel a discrete normal subgroup of G_0. Then, from the First Isomorphism Theorem for Lie groups (Theorem 4.6) $H \cong G_0/\ker(\Phi) = G_0/D$. □

So, in order to classify all the connected Lie groups that correspond to the Lie algebra $\mathfrak{g}$, we need to find the discrete normal subgroups of the simply-connected Lie group G_0. The result below will make that process easier.

Definition 5.6 (**Center of a Group**)**.** The set of elements of a group G that commute with all the elements of the group is called the **center of the group** and is denoted $Z(G)$. The center is a subgroup that is also normal. Every subgroup of the center is called **central subgroup**.

Theorem 5.7. *Every discrete normal subgroup of a connected Lie group is central.*

Proof. Let G be the connected Lie group and H a discrete normal subgroup. Then taking an arbitrary $h \in H$, we consider the map $\phi : G \to H$, $g \mapsto ghg^{-1}h^{-1}$. It is well defined, because from the normality property $ghg^{-1} \in H$, so $ghg^{-1}h^{-1} \in H$ as well. Since multiplication is smooth, then ϕ is also smooth. However, G is connected, so the image of G is also connected and, since H is discrete, the image must be a point. Plugging e to ϕ, we get that $\phi(e) = e$, so $\phi(g) = e$, $\forall g \in G$. Therefore, $gh = hg$, $\forall g$, so h belongs to the center of G. Since h is arbitrary, all elements of H belong to the center, so H is central. □

At last, we can state a process for finding all the connected Lie groups that have the same Lie algebra. Starting with the Lie algebra $\mathfrak{g}$, we find the corresponding simply-connected Lie group G_0. Then, we can calculate the center $Z(G_0)$ and take all its discrete subgroups D_i (which will also be normal). Finally, we take all the Lie groups G_0/D_i, which have the same Lie algebra $\mathfrak{g}$ and, from Proposition 3.11, have fundamental group $\pi_1(G_0/D_i) \cong D_i$.

6 $SU(2)$ is the universal covering group of $SO(3)$

Finally, we apply the results of the previous section to prove that the Lie group $SU(2)$ is a universal covering of the Lie group $SO(3)$ and to find the fundamental group of $SO(3)$.

First, in order to use the results of the previous sections, we need connectedness for these groups:

Proposition 6.1. *The groups $SO(n)$ are path-connected.*

Proof. All orthogonal matrices are diagonalisable. So, we can write some $A \in SO(3)$, as $A = VDV^T$, where D is a diagonal matrix. Since A is orthogonal $AA^T = I$, so all its eigenvalues have norm 1. Also, A is a real matrix, so for every complex eigenvalue, its conjugate is also an eigenvalue. It is known that a diagonal matrix D with complex entries is similar with a matrix replacing its pair of complex eigenvalues:

$$\begin{bmatrix} x+yi & 0 \\ 0 & x-yi \end{bmatrix} \sim \begin{bmatrix} x & y \\ -y & x \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

since $x^2 + y^2 = 1$.

So, we have made a new matrix D', such that $A = V'D'V'^T$. We construct a path $A(t) = V'D'(t)V'^T$, where in $D'(t)$ we have replaced every angle θ with $(1-t)\theta$. This path is continuous, it is inside $SO(n)$ and $A(1) = I$, so every matrix A has a continuous path to the identity, thus $SO(n)$ is connected. ☐

Proposition 6.2. *The groups $SU(n)$ are path-connected.*

Proof. As in the previous proof, we can write some matrix $A \in SU(n)$ as VDV^*, where D is a diagonal matrix. All the eigenvalues of A have norm 1, so they have the form $e^{i\theta}$. Therefore, we can construct a path $A(t) = VD(t)V^*$, where in $D(t)$ we have replaced every angle θ with $(1-t)\theta$. This path is continuous, it is inside $SU(n)$ and $A(1) = I$, so every matrix A has a continuous path to the identity, thus $SU(n)$ is connected. ☐

Proposition 6.3. *$SU(2)$ is a simply connected Lie group.*

Proof. From the definition of $SU(2)$, we can see that every matrix in $SU(2)$ has the form:

$$\begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. Hence, if $a = a_1 + ia_2$ and $b = b_1 + ib_2$, we have that $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$. Therefore, we get an isomorphism $SU(2) \cong S^3$.

In Theorem 2.12, we have showed that $\pi_1(S^3)$ is trivial, so $\pi_1(SU(2))$ is also trivial and $SU(2)$ is simply connected. □

Now we prove the isomorphism between the Lie algebras of $SU(2)$ and $SO(3)$ to show the covering property.

Proposition 6.4. *The Lie algebras* $\mathfrak{so}(3)$ *and* $\mathfrak{su}(2)$ *are isomorphic.*

Proof. The Lie algebra of a Lie group is isomorphic with the tangent space of the Lie group in the identity element. So, to compute the Lie algebra of $SO(3)$, we take an arbitrary smooth curve $c(t)$ with $c(0) = I$ and $c'(0) = X$. We have that $c(t)c(t)^T = I$, so differentiating this relation we get: $c'(t)c(t)^T + c(t)c'(t)^T = 0$. Thus, plugging $t = 0$, we get $X + X^T = 0$. Therefore,

$$\mathfrak{so}(3) = \{X \in GL(3, \mathbb{R}) : X + X^T = 0\}$$

Solving this equation, we get that all elements X of $\mathfrak{so}(3)$ have the form:

$$\begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix}$$

So, we have the basis $\{L_x, L_y, L_z\}$ of $so(3)$ where

$$L_x = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad L_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad L_z \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

With the usual Lie bracket for matrices, we have $[L_x, L_y] = L_z, [L_z, L_x] = L_y, [L_y, L_z] = L_x$. Finally, $\mathfrak{so}(3) = span_{\mathbb{R}}(L_x, L_y, L_z)$.

Similarly, to compute the Lie algebra of $SU(2)$, we take an arbitrary smooth curve $c(t)$ with $c(0) = I$ and $c'(0) = X$. We have that $c(t)c(t)^* = I$, so differentiating this relation we get: $c'(t)c(t)^* + c(t)c'(t)^* = 0$. Thus, plugging $t = 0$, we get $X + X^* = 0$. Therefore,

$$\mathfrak{su}(2) = \{X \in GL(2, \mathbb{C}) : X + X^* = 0\}$$

Solving this equation, we get that all elements X of $\mathfrak{su}(2)$ have the form:

$$\begin{bmatrix} ai & -b + ci \\ b + ci & -ai \end{bmatrix}$$

where a, b, c are reals. So, we have the basis $\{u_1, u_2, u_3\}$ of $su(2)$ where

$$u_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad u_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad u_3 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

With the usual Lie bracket for matrices, we have $[u_1, u_2] = u_3, [u_3, u_1] = u_2, [u_2, u_3] = u_1$. Finally, $\mathfrak{su}(2) = span_{\mathbb{R}}(u_1, u_2, u_3)$.

Therefore, defining the homomorphism of 3-dimensional spaces $\phi : \mathfrak{so}(3) \to \mathfrak{su}(2)$, such that $\phi(L_x) = u_1$, $\phi(L_y) = u_2$ and $\phi(L_z) = u_3$, we see that it is a Lie algebra isomorphism, because L_x, L_y, L_z satisfy the same Lie bracket relations as its images. □

Now, that $SU(2)$ and $SO(3)$ have isomorphic Lie algebras and $SU(2)$ is simply-connected, we derive that $SU(2)$ is the universal covering group of $SO(3)$. Hence, we get the following results:

Proposition 6.5. *$SU(2)/\mathbb{Z}_2 \cong SO(3)$.*

Proof. First, we will find the center of the group $SU(2)$. Let $A \in Z(SU(2))$ such that

$$A = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}$$

Then it commutes with every $B \in SU(2)$. Taking

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

we get

$$\begin{bmatrix} b^* & a \\ -a^* & b \end{bmatrix} = \begin{bmatrix} b & a^* \\ -a & b^* \end{bmatrix}$$

concluding that a, b are real. Also it commutes with

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

hence taking the commutation relation

$$\begin{bmatrix} -bi & ai \\ ai & bi \end{bmatrix} = \begin{bmatrix} bi & ai \\ ai & -bi \end{bmatrix}$$

so $b = 1$ and $a = 1$ or $a = -1$, since $|a|^2 + |b|^2 = 1$. Therefore, we have that $Z(SU(2)) = \{I, -I\}$ and the normal subgroups are $\{I\}$ and $\{I, -I\}$.

Since the Lie algebras $su(2) \cong so(3)$, from the Theorem 5.5, we have that either $SO(3) \cong SU(2)$ or $SO(3) \cong SU(2)/\{I, -I\}$. We will show that $SO(3)$ and $SU(2)$ cannot be isomorphic with contradiction.

Assume that there is an isomorphism $\phi : SU(2) \to SO(3)$. Then, $\phi(I_2) = I_3$. We take the 3 matrices, which are elements of $SO(3)$:

$$D_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad D_B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad D_C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These 3 matrices squared are equal with the identity. So, their inverse images squared are also equal with the identity $(\phi^{-1}(D_A))^2 = \phi^{-1}(I_3) = I_2)$. Squaring a matrix in $SU(2)$ of the usual form we take that, if their squares equal identity, then they have the form:

$$\phi^{-1}(D_A) = \begin{bmatrix} a & 0 \\ 0 & a^* \end{bmatrix} \quad \phi^{-1}(D_B) = \begin{bmatrix} b & 0 \\ 0 & b^* \end{bmatrix} \quad \phi^{-1}(D_C) = \begin{bmatrix} c & 0 \\ 0 & c^* \end{bmatrix}$$

where $|a| = |b| = |c| = 1$. From the relations $D_A D_B = D_C$, $D_B D_C = D_A$ and $D_A D_C = D_B$, we take $ab = c$, $bc = a$ and $ac = b$. Reducing $a = e^{i\theta_A}, b = e^{i\theta_B}, c = e^{i\theta_C}$, we have that for all angles $\theta = \pm\pi$. So $a, b, c = \pm 1$.

But D_A, D_B, D_C are different matrices and two of them have equal inverse images, so ϕ^{-1} is not injective and ϕ is not an isomorphism.

As a result, we get that necessarily $SU(2)/\mathbb{Z}_2 \cong SO(3)$. □

Corollary 6.6. *The fundamental group of $SO(3)$ is isomorphic with $\mathbb{Z}_2$.*

Proof. If $\pi : SU(2) \to SO(3)$ is the covering map, then $\pi^{-1}(I_3) = \{I_2, -I_2\} \cong \mathbb{Z}_2$, since $SO(3) \cong SU(2)/\{I, -I\}$. So, from Proposition 3.11, $\pi_1(SO(3)) \cong \pi^{-1}(I_3) \cong \mathbb{Z}_2$. □

References

[1] Michael Patrick Cohen. *Lie Groups Lecture Notes.*

[2] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations.* Springer US, 2004.

[3] Allen Hatcher. *Algebraic Topology.* Algebraic Topology. Cambridge University Press, 2002.

[4] J.M. Lee. *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics. Springer, 2003.

[5] L. I. Nicolaescu. *Lectures on the Geometry of Manifolds.* www.nd.edu/~lnicolae/Lectures.pdf, 2006.